3.1

The Fourier transform is given by

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}$$

(a) $x(n)=a^n$ for $n>0$ and = 0 otherwise

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T} = \sum_{n=0}^{\infty} a^{nT}e^{-jn\omega T} = \sum_{n=0}^{\infty} (a^T e^{-j\omega T})^n$$

It is sum of a serial, the convegence is given by Abel theorem:

(i) $|a^T e^{-j\omega T}| < 1$, i.e. $|a^T| < 1$, $\sum_{n=0}^{\infty} (a^T a^{-j\omega T})^n$ is convergent

$$X(e^{j\omega T}) = \frac{1}{1 - a^T e^{-j\omega T}}.$$

(ii) $|a^T e^{-j\omega T}| > 1$, or $|a^T| > 1$, $\sum_{n=0}^{\infty} (a^T a^{-j\omega T})^n$ is not convergent.

(iii) $|a^T e^{-j\omega T}| = 1$, the convergence of $\sum_{n=0}^{\infty} (a^T a^{-j\omega T})^n$ is uncertain.

(b) $x(n) = -a^n$ for n<0 and = 0 otherwise.

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T} = \sum_{n=-\infty}^{-1} -a^{nT}e^{-jn\omega T} = \sum_{n=-\infty}^{-1} (a^T e^{-j\omega T})^n$$

$$= \sum_{n=1}^{\infty} (a^{-T} e^{j\omega T})^n$$

By using Abel theorem, we have

(i) $|a^{-T} e^{j\omega T}| < 1$, or $|a^T| > 1$, $-\sum_{n=1}^{\infty} (a^{-T} e^{j\omega T})^n$ is convergent. and

$$X(e^{j\omega T}) = \frac{a^{-T} e^{j\omega T}}{1 - a^{-T} e^{j\omega T}}.$$

(ii) $|a^{-T} e^{j\omega T}| > 1$, i.e. $|a^T| < 1$, $-\sum_{n=1}^{\infty} (a^{-T} e^{j\omega T})^n$ is not convergent.

(iii) $\left|a^{-T}e^{j\omega T}\right| = 1$, the convergence of $-\sum_{n=1}^{\infty}(a^{-T}e^{j\omega T})^n$ is uncertain.

3.2

Apply the definition of z-transform to the sequences using $Z\{u(n)\} = \dfrac{z}{z-1}$

(a) $x_1(nT)$

The period is 6, and the z-transform is

$$Z\{x_1(nT)\} = \sum_{n=-\infty}^{\infty} x_1(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty}(0z^0 - az^{-1} + 0z^{-2} + az^{-3} + 0z^{-4} - az^{-5})z^{-6n}$$

$$= \sum_{n=0}^{\infty}(-az^{-1} + az^{-3} - az^{-5})z^{-6n} = (-az^{-1} + az^{-3} - az^{-5})\sum_{n=0}^{\infty} z^{-6n}$$

$$= \frac{-a(z^5 - z^3 + z)}{z^6 - 1}$$

(b) $x_2(nT)$:

The period is 7, and the z-transform is

$$Z\{x_2(nT)\} = \sum_{n=-\infty}^{\infty} x_2(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty}(0az^0 + az^{-1} - az^{-2} - az^{-3} + az^{-4} + az^{-5} - az^{-6})z^{-7n}$$

$$= (az^0 + az^{-1} - az^{-2} - az^{-3} + az^{-4} + az^{-5} - az^{-6})\sum_{n=0}^{\infty} z^{-7n}$$

$$= a(1 + z^{-1} - z^{-2} - z^{-3} + z^{-4} + z^{-5} - z^{-6})\frac{z^7}{z^7 - 1}$$

$$= \frac{a(z^7 + z^6 - z^5 - z^4 + z^3 + z^2 - z^1)}{z^7 - 1}$$

(c) $x_3(nT)$

The period is 7, and the z-transform is

$$Z\{x_3(nT)\} = \sum_{n=-\infty}^{\infty} x_3(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^0 + 0z^{--1} - az^{-2} - 0z^{-3} + az^{-4} + 0z^{-5} - az^{-6})z^{-7n}$$

$$= (az^0 - az^{-2} + az^{-4} - az^{-6}) \sum_{n=0}^{\infty} z^{-7n} = \frac{a(z^7 - z^5 + z^3 - z^1)}{z^7 - 1}$$

(d) $x_4(nT)$

The period is 7, and the z-transform is

$$Z\{x_4(nT)\} = \sum_{n=-\infty}^{\infty} x_4(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^0 + 0z^{-1} - az^{-2} - az^{-3} + 0z^{-4} + az^{-5} - 0z^{-6})z^{-7n}$$

$$= (az^0 - az^{-2} - az^{-3} - az^{-5}) \sum_{n=0}^{\infty} z^{-7n} = \frac{a(z^7 - z^5 - z^4 + z^2)}{z^7 - 1}$$

(e) $x_5(nT)$:

The period is 7, and the z-transform is

$$Z\{x_5(nT)\} = \sum_{n=-\infty}^{\infty} x_5(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^0 + 2az^{-1} + az^{-2} - az^{-3} - 2az^{-4} - az^{-5} + 0z^{-6})z^{-7n}$$

$$= a(1 + 2z^{-1} + z^{-2} - z^{-3} - 2z^{-4} - z^{-5} + z^{-6}) \sum_{n=0}^{\infty} z^{-7n}$$

$$= \frac{a(z^7 + 2z^6 + z^5 - z^4 - 2z^3 - z^2 + z^1)}{z^7 - 1}$$

3.3 (a) $x(n\text{-}n_0) \leftrightarrow z^{-n_0}X(z)$

$$Z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-n} = \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-(n-n_0)-n_0}$$

$$= \sum_{n=-\infty}^{\infty} x(n-n_0)z^{-(n-n_0)}z^{-n_0} = z^{-n_0}\sum_{n=-\infty}^{\infty} x(n-n_0)z^{-(n-n_0)} = z^{-n_0}X(z)$$

(b) $a^n x(n) \leftrightarrow X\left(\frac{z}{a}\right)$

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

(c) $x(-n) \leftrightarrow X\left(\frac{1}{z}\right)$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n} = \sum_{m=-\infty}^{\infty} x(m)z^{m} = \sum_{m=-\infty}^{\infty} x(m)\left(\frac{1}{z}\right)^{-m} = X\left(\frac{1}{z}\right)$$

(d) $x^*(n) \leftrightarrow X^*(z^*)$

$$Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n)z^{-n} = \sum_{n=-\infty}^{\infty} x^*(n)((z^{-n})^*) = \sum_{n=-\infty}^{\infty} x^*(n)(z^{-n*})^*$$

$$= \sum_{n=-\infty}^{\infty} [x(n)(z^*)^{-n}]^* = X^*(z^*)$$

(Note: consider a complex number $z = ae^{i\theta}$, where $a = |z|$ and θ is the argument, $(z^n)^* = (a^n e^{in\theta})^* = (a^n e^{-in\theta})^* = (ae^{-i\theta})^n = (z^*)^n$,)

(e) $Re\{x(n)\} \leftrightarrow 0,5[X(z) + X^*(z^*)]$

$Re\{x(n)\} = 0,5[X(n) + x^*(n)]$

$$Z\{Re\{x(n)\}\} = \sum_{n=-\infty}^{\infty} 0,5[x(n) + x^*(n)]z^{-n} = 0,5\sum_{n=-\infty}^{\infty} [x(n)z^{-n} + x^*(n)z^{-n}]$$

$$= 0,5\left(\sum_{n=-\infty}^{\infty} x(n)z^{-n} + \sum_{n=-\infty}^{\infty} x^*(n)z^{-n}\right) = 0,5[X(z) + X^*(z^*)]$$

(f) $Im\{x(n)\} \leftrightarrow -0{,}5j[X(z) - X^*(z^*)]$

$Im\{x(n)\} = -0{,}5j[x(n) - x^*(n)]$, in the same manner as (e), we have

$Z\{Im\{x(n)\}\} = -0{,}5j[X(z) - X^*(z^*)]$.

Alternatively, we can derive it from (e),

since $x(n) = Re\{x(n)\} + j \cdot Im\{x(n)\}$, and $Z\{x(n)\} = X(z)$,

$Z\{Rex(n)\}\} = 0.5[X(z)+X^*(z^*)]$, with the linear property of z-transform,

$Z\{Im\{x(n)\}\} = \frac{1}{j}[Z\{x(n)-Re\{x(n)\}\}] = -0.5j[X(z)-X^*(z^*)]$.

3.4

(a) The autocorrelation function $r(k) = \sum_{n=0}^{\infty} x(n)x^*(n+k)(k \geq 0)$.

$$Z\{r(k)\} = \sum_{k=-\infty}^{\infty} r(k)z^{-k} = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} x(n)x^*(n+k)\right)z^{-k} =$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} x(n)x^*(n+k)z^{-k}\right) = \sum_{n=0}^{\infty} x(n)\left(\sum_{k=0}^{\infty} x^*(n+k)z^{-k}\right)$$

$$= \sum_{n=0}^{\infty} x(n)z^{n}[X^*(z^*)] = X\left(\frac{1}{z}\right)X^*(z^*)$$

(b) The convolution $y(n) = \sum_{n=0}^{\infty} h(k)x(n-k)$.

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} y(n)z^{-n} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} h(k)x(n-k)\right)z^{-n}$$

$$= \sum_{k=0}^{\infty} h(k)\left(\sum_{n=0}^{\infty} x(n-k)z^{-n}\right)$$

$$= \sum_{k=0}^{\infty} h(k)[z^{-k}X(z)] = X(z)H(z)$$

3.5 We need to find a closed formulae for T(n), i.e., for the t ime required to solve a large problem. We will do this by using the z - transform. However, we first make the substitution.

$$x(m) = \frac{T(c^m)}{b^m}$$

in order to obtain a linear difference equation

$$x(m) = \begin{cases} a & m = 0 \\ x(m-1) + \dfrac{dc^m}{b^m} & m > 0 \end{cases}$$

where we for the sake of simplicity have selected *MinSize* = 1. Further, we have assumed that the size of the subproblems is a power of c^m, i.e., the subproblems have sizes: n/c^1, n/c^2, n/c^3, etc. Applying the z-transform yields

$$X(z) = z^{-1}[X(z) + x(-1)z] + d\sum_{m=1}^{\infty}\left(\frac{c}{b}\right)^m z^{-m}$$

but the initial value, $m = 0$, yields $x(0) = x(-1) + d = a \Rightarrow$
$\Rightarrow x(-1) = a - d$

$$X(z) = z^{-1}X(z) + a - d + d\sum_{m=1}^{\infty}\left(\frac{c}{b}\right)^m z^{-m}$$

$$X(z) = \frac{z}{z-1}\left[a - d + \frac{dz}{z - \frac{c}{b}}\right]$$

$$= \frac{(a-d)z}{z-1} + \frac{d}{b-c}\left[\frac{bz}{z-1} - \frac{cz}{z-\frac{c}{b}}\right]$$

and

$$x(m) = a - d + \frac{bd}{b-c} - \frac{cd}{b-c}\left(\frac{c}{b}\right)^m = a - d + d\frac{1-\left(\frac{c}{b}\right)^{m+1}}{1-\frac{c}{b}} \Rightarrow$$

$$x(m) = a - d + d\sum_{i=0}^{m}\left(\frac{c}{b}\right)^i, m \ge 0$$

Thus

$$T(n) = T(c^m) = b^m\left[a - d + d\sum_{i=0}^{m}\left(\frac{c}{b}\right)^i\right] = (a-d)b^m + dc^m\sum_{i=0}^{m}\left(\frac{b}{c}\right)^i$$

but $m = \log_c(n)$.Finally, we get

$$T(n) = (a-d)b^{\log_c(n)} + dn\sum_{i=0}^{\log_c(n)}\left(\frac{b}{c}\right)^i.$$

We have three interesting cases.

Case: b < c

We get: $T(n) \in O\left[(a-d)b^{\log_c(n)} + dn\sum_{i=0}^{\log_c(n)}\left(\frac{b}{c}\right)^i\right]$

$$T(n) \in O[(a-d)b^{\log_c(n)}] + O\left[dn\sum_{i=0}^{\log_c(n)}\left(\frac{b}{c}\right)^i\right] = O[b^{\log_c(n)}] + O(n)$$

but we have $\lim_{n\to\infty}\frac{g(n)}{f(n)} = \lim_{n\to\infty}\frac{b^{\log_c(n)}}{n} = \lim_{n\to\infty}\frac{b^{\log_c(n)}\ln(b)}{n\ln(c)} =$

$$= \lim_{n\to\infty}\frac{b^{\log_c(n)}}{n}\frac{\ln(b)}{\ln(c)} = 0$$

Hence, $T(n) \in O(n)$ since, $b^{\log_c(n)}$ grows no faster than n.

Case: b = c

For b=c we have

$$T(n) = (a-d)b^m + dn\sum_{i=0}^{\log_c(n)} 1 = (a-d)n + dn(\log_c(n) + 1)$$

and

$$T(n) \in O[(a-d)n + dn(\log_c(n) + 1)] = O[n\log_c(n)]$$

since $\lim_{n \to \infty} \frac{n}{n \ln(n)} = 0$

Case: b > c

We have

$$T(n) = (a-d)b^m + dn \sum_{i=0}^{m} \left(\frac{b}{c}\right)^i = (a-d)b^m + dc^m \frac{1-\left(\frac{b}{c}\right)^{m+1}}{1-\frac{b}{c}}$$

$$T(n) \in O\left[(a-d)b^m + dc^m \frac{1-\left(\frac{b}{c}\right)^{m+1}}{1-\frac{b}{c}}\right] = O[b^m]$$

Finally we get: $T(n) \in O[b^{\log_c(n)}] = [n^{\log_c(b)}]$

3.9 A system is causal if and only if

$x_1(n) = x_2(n)$ for $n \le n_0$ if $y_1(n) = y_2(n)$ for $n \le n_0$

Now, an LSI system is described by the convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(n)h(n-k)$$

We must have $y_1(n_0) = y_2(n_0)$, =>

$$y_1(n_0) - y_2(n_0) = \sum_{k=-\infty}^{\infty} [x_1(k) - x_2(k)]h(n_0 - k) = \sum_{k=n_0}^{\infty} [x_1(k) - x_2(k)]h(n_0 - k)$$

since $x_1(k) = x_2(k)$ for $k \le n_0$.

Thus, $h(n_0 - k) = 0$ for $k \le n_0$ since all terms in the convolution must be zero, i.e., $h(n) = 0$ for $n < 0$. On the other hand, if $h(n) = 0$ for $n < 0$, we have

$$y(n_0) = \sum_{k=-\infty}^{\infty} x(n)h(n_0 - k) = \sum_{k=-\infty}^{n_0} x(n)h(n_0 - k)$$

=> $y(n)$ is independent of $x(k)$ for $k > n_0$, i.e., independent of the future input samples.

3.6 $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} = \sum_{n=-\infty}^{\infty} (0.8)^n z^{-n} - \sum_{n=-\infty}^{\infty} (0.6)^n z^{-n}$

$$= \frac{1}{1-0.8z^{-1}} - \frac{1}{1-0.6z^{-1}} = \frac{z}{z-0.8} - \frac{z}{z-0.6} =$$

$$= \frac{0.2z}{(z-0.8)(z-0.6)}$$

The geometric series converges for:

$|0.8\,z^{-1}| < 1$ and $|0.6\,z^{-1}| < 1$, respectively. We get:

$$H(z) = \frac{0.2z}{(z-0.8)(z-0.6)} \quad , |z| > 0.8$$

3.7 The step response is obtained by accumulating the impulse response values:

$$s(n) = \sum_{k=-\infty}^{\infty} h(k) = \sum_{k=0}^{\infty} (0.8)^k - \sum_{k=0}^{\infty} (0.6)^k = \frac{1-(0.8)^{n+1}}{1-0.8} - \frac{1-(0.6)^{n+1}}{1-0.6}$$

Note that $s(n) \to H(1)$ as $n \to \infty$.

Proof

$$s(n) = \sum_{k=-\infty}^{n} h(k) \to \sum_{k=-\infty}^{\infty} h(k)z^{-k} = H(1) \text{ for } z = 1.$$

In this case we have $s(n) \to \frac{1}{1-0.8} - \frac{1}{1-0.6} = 0.25$ and

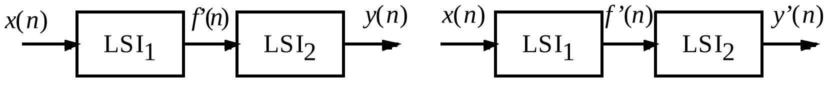
$$H(1) = \frac{0.2}{(1-0.8)(1-0.6)} = \frac{0.2}{0.2 \times 0.4} = 0.25$$

3.8 The passband for the digital filter should be: f_c = 25 kHz. Hence, the next image of the passband starts at: $f_{sample} - f_c$. From a filter table, or *Delfi*, we find that a third-order Butterworth filter, with 1 dB in the passband, has an attenuation of 40 dB at

$6\,f_c = 150$ kHz, i.e., $(f_{sample} - f_c)/f_c = 6 \Rightarrow f_{sample} = 7 \cdot 25 = 175$ kHz

3.10

The different orderings of the two cascade LSI systems are shown below



(a) (b)

Assume that the transfer function LSI_1 is $L_1(z)$ and the transfer function of LSI_2 is $L_2(z)$.

For the system in (a), the z-transform of output $y(n)$ can be calculated by
$Y(z) = L_2(z)F(z) = L_2(z)L_1(z)X(z)$;
For the system in (b), the z-transform of output $y'(n)$ can be calculated by
$Y'(z) = L_1(z)F'(z) = L_1(z)L_2(z)X(z)$.
Since $L_1(z)L_2(z) = L_2(z)L_1(z)$ according to the communitive law of multiplication, we can state $Y(z) = Y'(z)$, i.e. the ordering of two cascade LSI systems may be interchanged.

3.11

The difference equation is $y(n) = by(n - 1) + ax(n)$. Apply z-transform to both sides of the equation we have

$$Y(z) = bz^{-1}Y(z) + aX(z)$$

which gives the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a}{1 - bz^{-1}}.$$

3.12a) $$H(z) = \frac{1,2z + 1,2}{z^2 - 1,6z + 0,63} = \frac{11,4}{z - 0,9} - \frac{10,2}{z - 0,7}$$

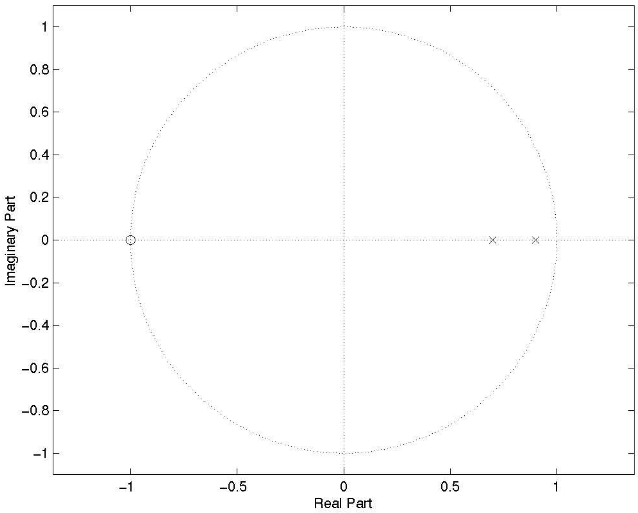
$$= \frac{11,4}{0,9}\sum^{\infty} 0,9^n z^{-n} - \frac{10,2}{0,7}\sum^{\infty} 0,7^n z^{-n}$$

By identification we get

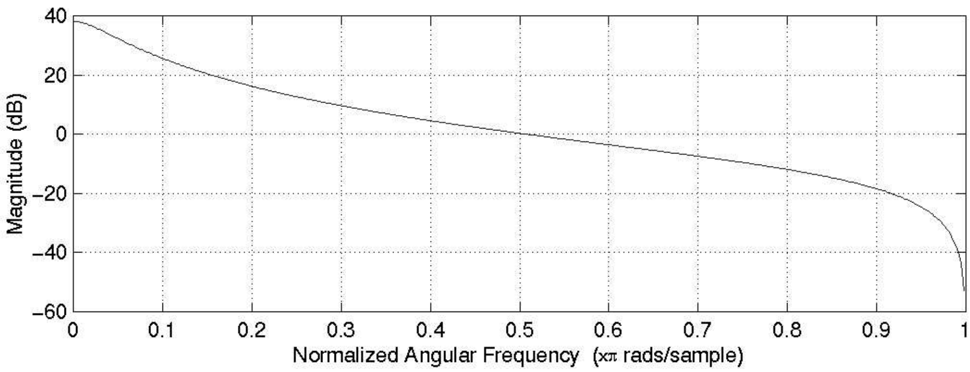
$$h(n) = \begin{cases} 0 & n < 1 \\ 11,4 \times 0,9^{n-1} - 10,2 \times 0,7^{n-1} & n \geq 1 \end{cases}$$

b) The region of convergence for the two geometric series are $|z| > 0,9$ and $|z| > 0,7$, respectively. The region of convergence for $H(z)$ is where both series convergences, i.e., $|z| > 0,9$.

c) The poles are $z = 0{,}9$ and $z = 0{,}7$ and the only zero is $z = -1$.



d) The magnitude is plotted in the following figure



e) The step response ($n > 0$) is

$$s(n) = \sum_{k=1}^{n} h(k) = 11{,}4 \times \sum_{k=1}^{n} 0{,}9^{k-1} - 10{,}2 \times \sum_{k=1}^{n} 0{,}7^{k-1}$$

$$= 11{,}4 \times \left(\frac{1 - 0{,}9^n}{1 - 0{,}9}\right) - 10{,}2 \times \left(\frac{1 - 0{,}7^n}{1 - 0{,}7}\right)$$

$$= 80 - 114 \times 0{,}9^n + 34 \times 0{,}7^n$$

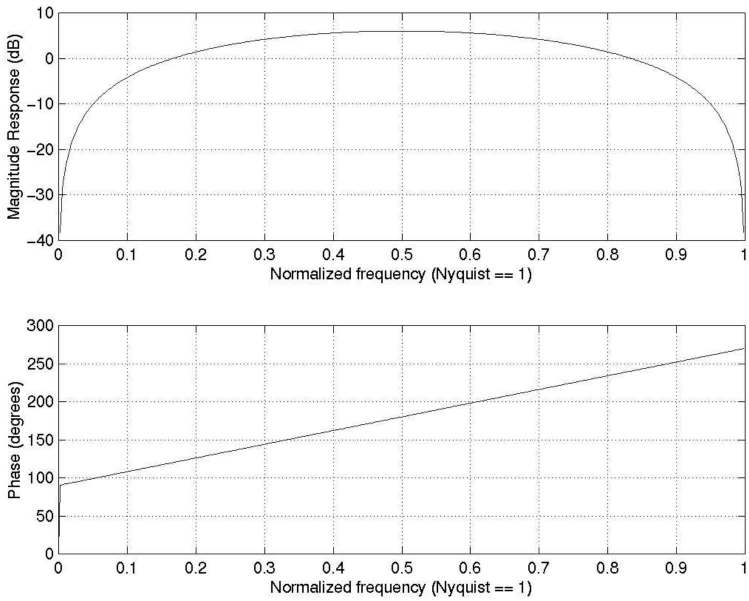
3.13

The difference equation is $y(n) = -by(n-2) + a[x(n) - x(n-2)]$. Using the z-transform and the definition of transfer function, the transfer, the trasnfer function is

$$H(z) = \frac{a - az^{-2}}{1 + bz^{-2}}$$

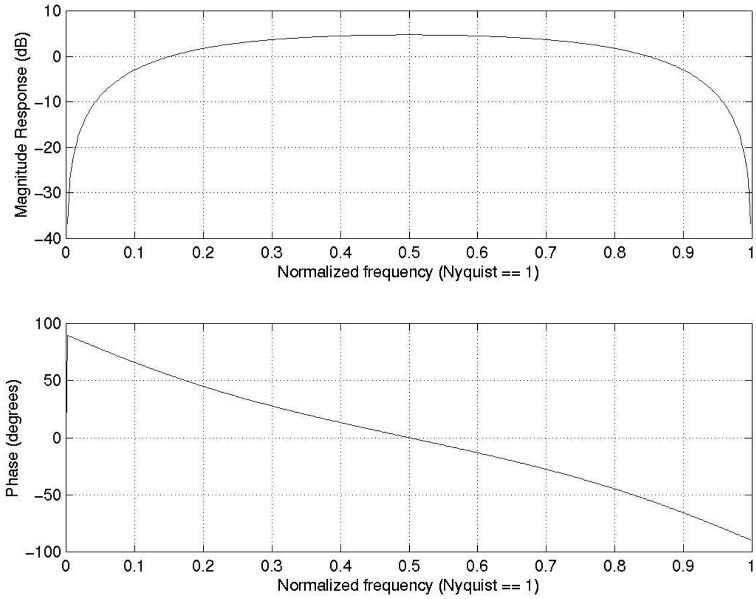
The transfer function has two zeros in ± 1. The placement of poles are determined by the value of *b*.

If $b = 0$, $H(z) = a(1 - z^{-2})$, with $a = 1$, we can plot the magnitude and phase responses for the filter with *Matlab*™. the groupdelay is -¼ except at the normalized frequency 0 and 1, which means that this linear filter has linear phase.



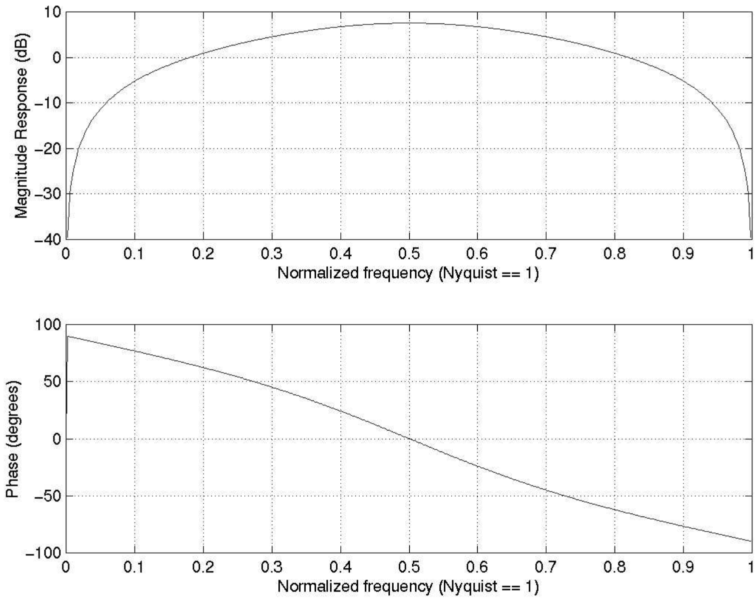
Transfer function plot (b=0)

If $b<0$, the poles are $\pm\sqrt{-b}$, when $b>-1$, the filter is stable, with $a = 1$ and $b = -0.16$, we can plot the magnitude and phase responses as follow



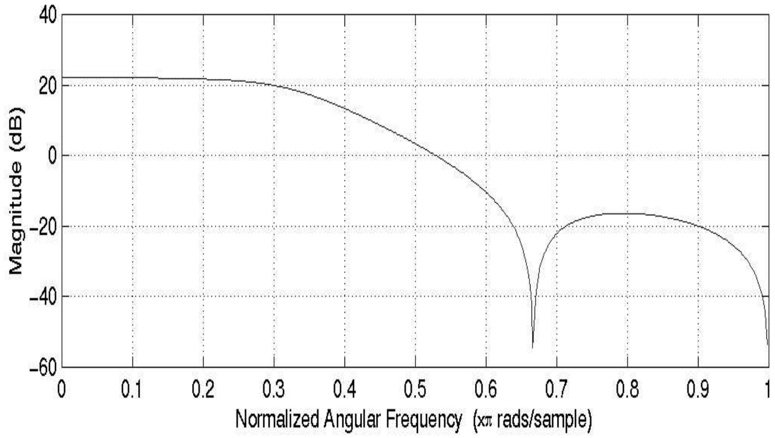
Transfer function plot ($-1<b<0$).

If b>9, there are poles at $\pm j\sqrt{b}$, when $1>b>0$, the filter is stable, with a = 1 and b = 0.16, we plot the magnitude and phase as the figure below

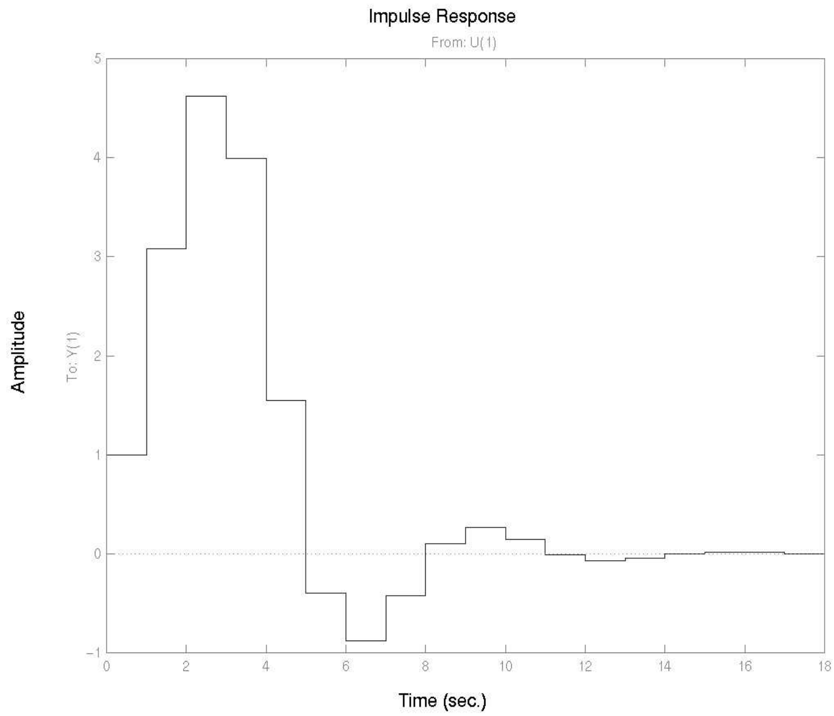


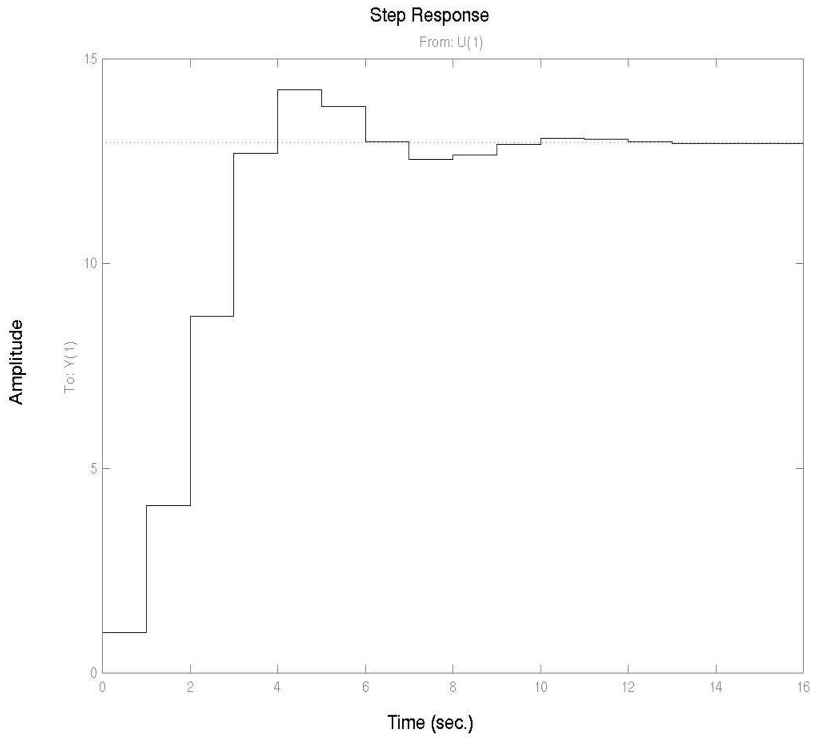
Transfer function plot ($0<b<1$)

3.14 a)The filter is a lowpass filter with a cutoff angle of about 60° and a stopband angle of about 100°.



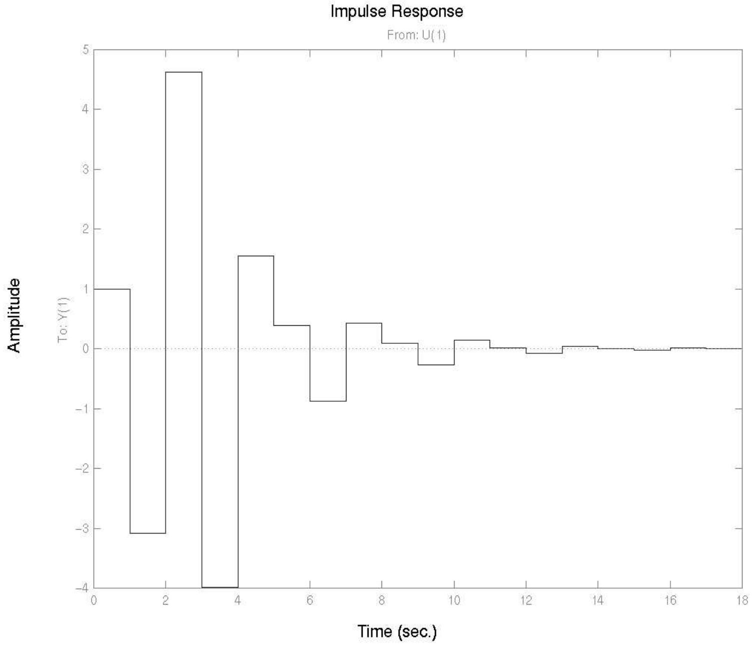
b)

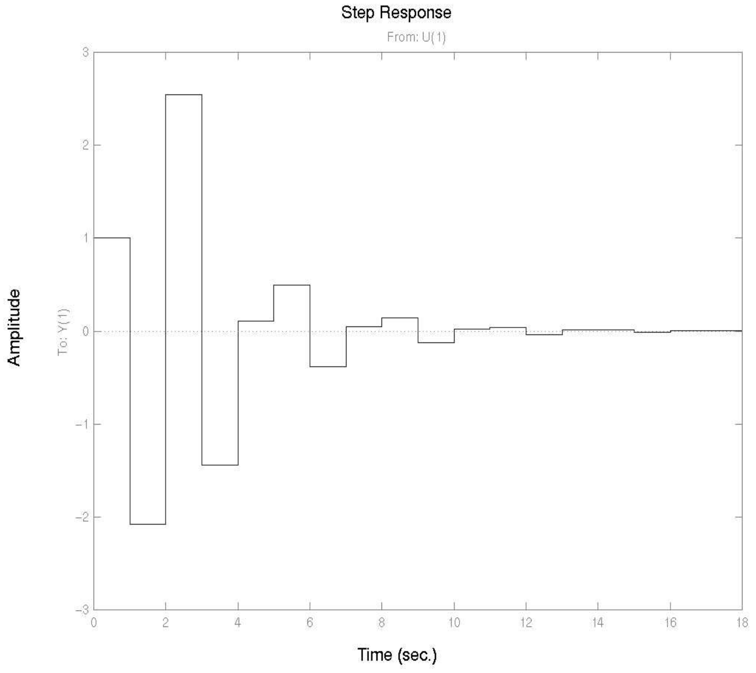




c) Changing the sign of the *a*- and *b*-coefficients in the second-order sections corresponds the changing the sign of real part of the poles and zeros. The pole-zero configuration is mirrored in the imaginary axis. Hence, the filter becomes a highpass filter. The new impulse response is

$h_{new}(n) = h_{old}(n)\,(-1)^n$





3.15

Prove: Since $z = e^{j\omega t}$, or $\omega = \frac{1}{jT}\ln z$, we change the derivation basis from $\frac{\partial}{\partial z}$ to $\frac{\partial}{\partial \omega}$

by $\frac{\partial}{\partial z} = \frac{d\omega}{dz} \cdot \frac{\partial}{\partial \omega} = \frac{d\left(\frac{1}{jT}\ln z\right)}{dz} \cdot \frac{\partial}{\partial \omega} = \frac{1}{jTz} \cdot \frac{\partial}{\partial \omega} = \frac{je^{-j\omega T}}{T} \cdot \frac{\partial}{\partial \omega}$.

The transfer function can be written as

$$H(z) = |H(z)|e^{j\Phi(z)}$$

The phase function is therefore $\Phi(z) = -j[\ln H(z) - \ln|H(z)|]$.

Compute the $z\frac{d}{dz}\ln H(z)$, we have

$$z\frac{d}{dz}\ln H(z) = z\frac{d}{dz}\ln[|H(z)|e^{j\Phi(z)}] = z\left\{\frac{d}{dz}\ln|H(z)| + j\frac{d}{dz}\Phi(z)\right\}$$

$$= z\left\{\frac{\frac{d}{dz}|H(z)|}{|H(z)|} + j\frac{d}{dz}\Phi(z)\right\}$$

Change the derivation basis,

$$z\frac{d}{dz}\ln H(z) = \frac{z}{|H(z)|}\left\{\frac{d}{dz}|H(z)| + j|H(z)|\frac{d}{dz}\Phi(z)\right\}$$

$$= \frac{e^{j\omega T}}{|H(e^{j\omega T})|}\left\{\frac{je^{-j\omega T}}{T}\frac{\partial}{\partial\omega}|H(e^{j\omega T})| + \frac{e^{-j\omega T}}{T}|H(e^{j\omega T})|\frac{\partial}{\partial\omega}\Phi(e^{j\omega T})\right\}$$

$$= \frac{1}{|H(e^{j\omega T})|T}\left\{|H(e^{j\omega T})|\frac{\partial}{\partial\omega}\Phi(e^{j\omega T}) - j\frac{\partial}{\partial\omega}|H(e^{j\omega T})|\right\}$$

Since $|H(e^{j\omega T})|$ and $\Phi(e^{j\omega T})$ are real functions, $\frac{\partial}{\partial\omega}|H(e^{j\omega T})|$ and $\frac{\partial}{\partial\omega}\Phi(e^{j\omega T})$ are therefore real functions, which gives that

$$Re\left\{z\frac{d}{dz}\ln H(z)\right\} = Re\left\{\frac{1}{|H(e^{j\omega T})|T}|H(e^{j\omega T})|\frac{\partial}{\partial\omega}\Phi(e^{j\omega T}) - j\frac{\partial}{\partial\omega}|H(e^{j\omega T})|\right\}$$

$$= \frac{1}{|H(e^{j\omega T})|T}\left\{|H(e^{j\omega T})|\frac{\partial}{\partial\omega}\Phi(e^{j\omega T})\right\} = \frac{1}{T}\frac{\partial}{\partial\omega}\Phi(e^{j\omega T}).$$

Compare with the defination of group delay, we have

$$\tau_g(\omega T) = -\frac{\partial}{\partial\omega}\Phi(\omega T) = -TRe\left\{z\frac{d}{dz}\ln H(z)\right\} \text{ for } z = e^{j\omega T} .$$

3.16In an electrocardiogram (E.C.G.) measurement, the form of a curve is important. The linearity of phase, which is predicted by group delay, is therefore a major interest. If the group delay is over a limit, the form of curve is deformed which could lead to wrong judgement.

3.17 We have $|H(e^{j\omega T})|^2 = H(e^{j\omega T})H^*(e^{j\omega T}) =$

$$= \frac{1 - ae^{j\omega T}}{e^{j\omega T} - a}\frac{1 - ae^{-j\omega T}}{e^{-j\omega T} - a} = \frac{1 - ae^{-j\omega T} - ae^{j\omega T} + a^2}{1 - ae^{j\omega T} - ae^{-j\omega T} + a^2} = 1$$

3.18

Using the notations of signals in the following figure, the difference equation can be written as:

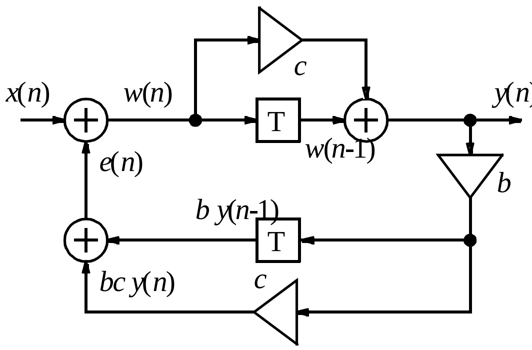
$e(n) = b(cy(n) + y(n - 1))$

$w(n) = e(n) + x(n) = b(cy(n) + y(n - 1)) + x(n)$

$y(n) = aw(n) + w(n-1)$

$= ab(cy(n) + y(n -1)) + ax(n) + b(cy(n -1) + y(n - 2)) + x(n - 1)$

$= b(acy(n) + (a + c)y(n - 1) + y(n - 1) + y(n - 2)) + ax(n) + x(n - 1)$



Apply the z-transform to the difference equation, the transfer function is thereby

$$H(z) = \frac{a + z^{-1}}{(1 - abc) - b(a + c)z^{-1} - bz^{-2}}$$

The zero is -a, and the poles are $\dfrac{-b(a + c) \pm \sqrt{b(ba^2 - 2abc + bc^2 + 4)}}{2(-1 + abc)}$. The sketch of polezero configuration is left to the reader to exploit the value combinations of a, b and c.

3.19 We have: $X(k) = \sum_{n=0}^{N-1} x(n)W^{nk} = \sum_{n=0}^{N-1} x^*(n)W^{nk} =$

$$= \left[\sum_{n=0}^{N-1} x(n)W^{-nk}\right]^* = \left[\sum_{n=0}^{N-1} x(n)W^{(N-k)n}\right]^* = X^*(N-k).$$

Thus, $X(k) = X^*(N-k)$

3.22

Show that $\sum_{n=0}^{N-1} W^{kn} = \begin{cases} N & k = 0 \\ 0 & \text{otherwise} \end{cases}$

where $W = e^{-i2\pi/N}$.

(i) If $k = 0$, $\sum_{n=0}^{N-1} W^{kn} = \sum_{n=0}^{N-1} W^{0} = \sum_{n=0}^{N-1} 1 = N$.

(ii)If $k ¦ 0$, $W^{k} = e^{-i2\pi k/N} \neq 1$,

$$\sum_{n=0}^{N-1} W^{kn} = \frac{1 - W^{kN}}{1 - W^{k}} = \frac{1 - (e^{-i2\pi/N})^{kN}}{1 - e^{-i2\pi k/N}} = \frac{1 - e^{-i2\pi k}}{1 - e^{-i2\pi k/N}}$$

$$= \frac{1 - 1}{1 - e^{-i2\pi k/N}} = 0$$

With the results for (i) and (ii), we have shown that

$$\sum_{n=0}^{N-1} W^{kn} = \begin{cases} N & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

3.23

DFT with transform length $N = 8$

$$\underline{\bar{x}}(k) = \sum_{n=0}^{7} x(n) W^{nk} = \sum_{n=0}^{3} x(2n) W^{2nk} + \sum_{n=0}^{3} x(2n+1) W^{(2n+1)k}$$

$$= \sum_{n=0}^{3} x(2n) W^{(2n)k} + W^{k} \sum_{n=0}^{3} x(2n+1) W^{2nk}$$

for $k = 0, 1, \ldots, 7$

Now, $W^{2nk} = e^{-jn\pi(2nk)/8} = e^{-jn\pi(nk)/4} = W'^{nk}$

where $W' = e^{-j2\pi/4}$. Hence, the DFT of length 8 can be partitioned into two DFTs of length 4, the computation can now be partitioned by

$\underline{\bar{x}}(k) = DFT\{x(zn)\} + W^{k} DFT\{x(zn+1)\}$

$= DFT\{x_{\text{even}}(n')\} + W^{k} DFT\{x_{\text{odds}}(n')\}$

where $n' = 0, 1, 2, 3$, and $x_{\text{even}}(n') = x(2n')$.

$x_{\text{odds}}(n') = x(2n' + 1)$.

3.24

Since it requires multiplications before additions, it uses decimations-in-time FFT. From the algorithm description in Box 3.3, we can identify the twiddle factor in figure 3.22:

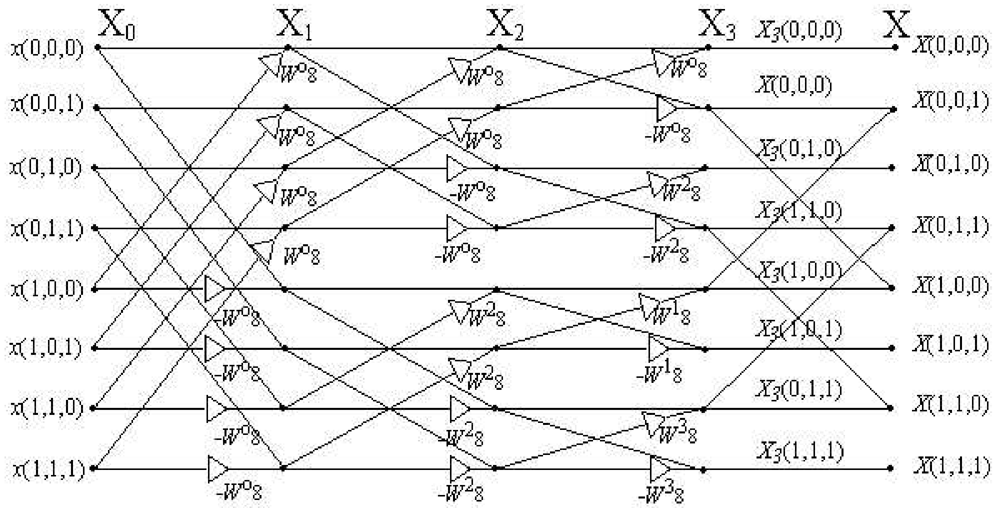


Fig 3.22

The minus signs in the SFG mean the substraction in butterfly operations

The constant in the Box should be

$N = 8$, $M = 3$, N_{minus} 1 = 7 now.3

At stage 1:

$N_S = 4$,

Table 1:

q	k	block_index	ρ	complex constant
1	0	0	0	$W^0{}_8$
2	1	0	0	$W^0{}_8$
3	2	0	0	$W^0{}_8$
4	3	0	0	$W^0{}_8$

At stage 2: $N_S = 2$,

q	k	block_index	ρ	complex constant
1	0	0	0	$W^0{}_8$
2	1	0	0	$W^0{}_8$
1	4	1	2	$W^2{}_8$

q	k	block_index	ρ	complex constant
2	5	1	2	$W^2{}_8$

At stage 3:
$N_s = 1$,

q	k	block_index	ρ	complex constant
1	0	0	0	$W^0{}_8$
1	2	1	2	$W^2{}_8$
1	4	2	1	$W^1{}_8$
1	6	3	3	$W^3{}_8$

3.26 Let $X(n) = A + jB$ Thus, we have obtained the IFFT except of the factor 1N . However, this factor will, in practice, be included inside the butterflies in order to achieve a safe scaling of the signals. and $W^{nk} = W_R + jW_I$. According to the suggested algorithm we start by computing the DFT of X(k). Hence, we compute the DFT where we have interchanged the real and imaginary parts of discrete Fourier transform, $X(k)$

$$\sum_{n=0}^{N-1} X(n)W^{nk} = \sum_{n=0}^{N-1} (A + jB)(W_R + jW_I) =$$

$$= \sum_{n=0}^{N-1} (BW_R + jBW_I + jAW_R - AW_I) =$$

$$= \sum_{n=0}^{N-1} [(BW_R - AW_I) + j(BW_I + AW_R)] =$$

Now, we interchange the real and imaginary parts of the DFT

$$\sum_{n=0}^{N-1} [(BW_R - AW_I) + j(BW_I + AW_R)] =$$

$$= \sum_{n=0}^{N-1} (A + jB)(W_R - jW_I) =$$

$$= \sum_{n=0}^{N-1} (A + jB)W^{-nk} = \sum_{n=0}^{N-1} X(n)W^{-nk} = x(k)$$

Thus, we have obtained the IFFT except of the factor $1/N$. However, this factor will, in practice, be included inside the butterflies in order to achieve a safe scaling of the signals.

An alternative method is as follows.

$$x(k) = \frac{1}{N}\sum_{n=0}^{N-1} X(n)W^{-nk} = \frac{1}{N}\left[\sum_{n=0}^{N-1} (X(n)W^{-nk})^*\right]^* =$$

$$= \frac{1}{N}\left[\sum_{n=0}^{N-1} X^*(n)W^{nk}\right]^* = \frac{1}{N}[\mathrm{DFT}\{X^*(n)\}]^*$$

This alternative method to compute the IFFT by using two complex conjugate operations is more expensive than the method discussed above, since it involves two changes of the sign of the imaginary parts.

3.27 We form a new, complex-valued sequence from the two real-valued sequences.

$$z(n) = x(n) + j\,y(n)$$

The DFT is $Z(k) = \sum_{n=0}^{N-1} z(n)W^{nk} = \sum_{n=0}^{N-1} z(n)e^{-2\pi nk/N}$.

Now, we compute the complex conjugate of the rotated $Z(k)$ values

$$Z^*(N-k) = \sum_{n=0}^{N-1} [x(n) - jy(n)]e^{2\pi n(N-k)/N}$$

where

$$e^{2\pi n(N-k)/N} = e^{2\pi n}e^{-2\pi nk/N} = e^{-2\pi nk/N}.$$

Hence, we get $Z^*(N-k) = \sum_{n=0}^{N-1} [x(n) - jy(n)]e^{-2\pi nk/N}$.

Adding these two expressions we get:

$$Z(k) + Z^*(N-k) = \sum_{n=0}^{N-1} [x(n) + jy(n)]e^{-2\pi nk/N} + \sum_{n=0}^{N-1} [x(n) - jy(n)]e^{-2\pi nk/N}$$

$$= \sum_{n=0}^{N-1} 2x(n)e^{-2\pi nk/N}$$

Hence, $Z(k) + Z^*(N-k) = 2X(k)$ and $X(k) = 0.5[Z(k) + Z^*(N-k)]$

Similarly, subtracting the two expressions we get

$$Y(k) = -0.5\,j[Z(k) - Z^*(N-k)]$$

Hence, the DFTs of two real-valued sequences can be computed simultaneously without any significant additional cost. The inverse DFT can be computed in the same way.

3.29 $$X(k) = \sqrt{\frac{2}{N-1}} \sum_{n=0}^{N-1} c_k x(n) \cos\left(\frac{\pi n k}{N-1}\right) \qquad k = 0, 1, \ldots, N-1$$

where $c_k = \begin{cases} \frac{1}{2} & \text{for} \quad k = 0 \text{ or } k = N-1 \\ 1 & \text{for} \quad k = 1, 2, \ldots, N-2 \end{cases}$.

We can write the DCT in matrix form: $\boldsymbol{X}(k) = \sqrt{\frac{2}{N-1}} \boldsymbol{C}\boldsymbol{x}(n)$.

The rows of $\boldsymbol{C}$ is referred to as the basis vectors. Desirable properties for the DCT are:

1. The same expression for the DCT transform and the inverse transform. Hence, only one algorithm need to be implemented.
2. The application requires the DC component do not leak into other fre quency components. This is due to the high DC content in images. We can view the DCT as a set of FIR filters. Hence, the filers must have a zero at z = 1, except for the first lowpass filter.
3. The basis vectors should be symmetric or antisymmetric. This simplifies and reduces the hardware implementation.
4. The transform should be orthogonal in order to efficiently decorrelate the images.

All of this properties cannot be satisfied simultaneously. We therefore relaxes the orthogonality requirement slightly. The matrix $\boldsymbol{C}$ is

$$\begin{matrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\cos(0) & \cos(\frac{\pi}{7}) & \cos(\frac{2\pi}{7}) & \cos(\frac{3\pi}{7}) & \cos(\frac{4\pi}{7}) & \cos(\frac{5\pi}{7}) & \cos(\frac{6\pi}{7}) & -1 \\
\cos(0) & \cos(\frac{2\pi}{7}) & \cos(\frac{4\pi}{7}) & \cos(\frac{6\pi}{7}) & \cos(\frac{8\pi}{7}) & \cos(\frac{10\pi}{7}) & \cos(\frac{12\pi}{7}) & 1 \\
\cos(0) & \cos(\frac{3\pi}{7}) & \cos(\frac{6\pi}{7}) & \cos(\frac{9\pi}{7}) & \cos(\frac{12\pi}{7}) & \cos(\frac{15\pi}{7}) & \cos(\frac{18\pi}{7}) & -1 \\
\cos(0) & \cos(\frac{4\pi}{7}) & \cos(\frac{8\pi}{7}) & \cos(\frac{12\pi}{7}) & \cos(\frac{16\pi}{7}) & \cos(\frac{20\pi}{7}) & \cos(\frac{24\pi}{7}) & 1 \\
\cos(0) & \cos(\frac{5\pi}{7}) & \cos(\frac{10\pi}{7}) & \cos(\frac{15\pi}{7}) & \cos(\frac{20\pi}{7}) & \cos(\frac{25\pi}{7}) & \cos(\frac{30\pi}{7}) & -1 \\
\cos(0) & \cos(\frac{6\pi}{7}) & \cos(\frac{12\pi}{7}) & \cos(\frac{18\pi}{7}) & \cos(\frac{24\pi}{7}) & \cos(\frac{30\pi}{7}) & \cos(\frac{36\pi}{7}) & -1 \\
\frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2}
\end{matrix}$$

After simplification, we have:

$$\begin{matrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \cos(\frac{\pi}{7}) & \cos(\frac{2\pi}{7}) & \cos(\frac{3\pi}{7}) & -\cos(\frac{3\pi}{7}) & -\cos(\frac{2\pi}{7}) & -\cos(\frac{\pi}{7}) & -1 \\
1 & \cos(\frac{2\pi}{7}) & \cos(\frac{4\pi}{7}) & \cos(\frac{6\pi}{7}) & \cos(\frac{6\pi}{7}) & \cos(\frac{4\pi}{7}) & \cos(\frac{2\pi}{7}) & 1 \\
1 & \cos(\frac{3\pi}{7}) & \cos(\frac{6\pi}{7}) & \cos(\frac{9\pi}{7}) & -\cos(\frac{9\pi}{7}) & -\cos(\frac{6\pi}{7}) & -\cos(\frac{3\pi}{7}) & -1 \\
1 & \cos(\frac{4\pi}{7}) & \cos(\frac{8\pi}{7}) & \cos(\frac{12\pi}{7}) & \cos(\frac{12\pi}{7}) & \cos(\frac{8\pi}{7}) & \cos(\frac{4\pi}{7}) & 1 \\
1 & -\cos(\frac{5\pi}{7}) & \cos(\frac{10\pi}{7}) & \cos(\frac{15\pi}{7}) & -\cos(\frac{15\pi}{7}) & -\cos(\frac{10\pi}{7}) & -\cos(\frac{5\pi}{7}) & -1 \\
1 & -\cos(\frac{6\pi}{7}) & \cos(\frac{12\pi}{7}) & \cos(\frac{18\pi}{7}) & -\cos(\frac{18\pi}{7}) & -\cos(\frac{12\pi}{7}) & -\cos(\frac{6\pi}{7}) & -1 \\
\frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2}
\end{matrix}$$

3.30 PAL: $N = (720 \cdot 576 \cdot 8 + 2 \cdot 360 \cdot 576 \cdot 8) \cdot \frac{50}{2} = 166$ Mbit/s

NTSC: $N = (720 \cdot 480 \cdot 8 + 2 \cdot 360 \cdot 480 \cdot 8) \cdot \frac{59.94}{2} = 166$ Mbit/s

3.32 The sample frequency is 44.1 kHz. The resolution is given by the reciprocal of the FFT length. The length is

$$L = 1024 T_{sample} = \frac{1024}{44.1 \times 10^3}$$

We get $\Delta f = \frac{1}{L} = \frac{44.1 \times 10^3}{1024} = 43.07$ Hz

3.33 a) DCT–I ↔ SDCT
DCT–II ↔ EDCT
DCT–III^T (Transposed) ↔ DCT–II
DCT–IV is a shifted version of the SDCT

b) The relations derived i a) gives the kernel K for the inverse transforms.

IDCT–I	↔	$K_{DCT-I}{}^T = K_{DCT-I}$
IDCT–II	↔	$K_{DCT-II}{}^T = K_{DCT-III}$
IDCT–III	↔	$K_{DCT-III}{}^T = K_{DCT-II}$
IDCT–IV	↔	$K_{DCT-IV}{}^T = K_{DCT-IV}$