3.5 We need to find a closed formulae for $T(n)$, i.e., for the time required to solve a large problem. We will do this by using the z-transform. However, we first make the substitution.

$$x(m) = \frac{T(c^m)}{b^m}$$

in order to obtain a linear difference equation

$$x(m) = \begin{cases} a & m = 0 \\ x(m-1) + \dfrac{dc^m}{b^m} & m > 0 \end{cases}$$

where we for the sake of simplicity have selected $MinSize = 1$. Further, we have assumed that the size of the subproblems is a power of c^m, i.e., the subproblems have sizes: n/c^1, n/c^2, n/c^3, etc. Applying the z-transform yields

$$X(z) = z^{-1} [X(z) + x(-1)z] + d \sum_{m=1}^{\infty} (\frac{c}{b})^m z^{-m}$$

but the initial value, $m = 0$, yields $x(0) = x(-1) + d = a \Rightarrow$
$\Rightarrow x(-1) = a - d$

$$X(z) = z^{-1} X(z) + a - d + d \sum_{m=0}^{\infty} (\frac{c}{b})^m z^{-m}$$

$$X(z) = \frac{z}{z - 1} \left[a - d + \frac{d\ z}{z - \frac{c}{b}}\right] =$$

$$= \frac{(a - d)\ z}{z - 1} + \frac{d}{b - c} \left[\frac{b\ z}{z - 1} - \frac{c\ z}{z - \frac{c}{b}} \right]$$

and

$$x(m) = a - d + \frac{b\ d}{b - c} - \frac{c\ d}{b - c} (\frac{c}{b})^m = a - d + d \frac{1 - (\frac{c}{b})^{m+1}}{1 - \frac{c}{b}} \Rightarrow$$

$$x(m) = a - d + d \sum_{i=0}^{m} \left(\frac{c}{b}\right)^{i}, \quad m \geq 0$$

Thus

$$T(n) = T(c^m) = b^m \left[a - d + d \sum_{i=0}^{m} \left(\frac{c}{b}\right)^{i}\right] = (a-d)b^m + d\, c^m \sum_{i=0}^{m} \left(\frac{b}{c}\right)^{i}$$

but $m = log_c(n)$. Finally we get

$$T(n) = (a-d)b^{log_c(n)} + d\, n \sum_{i=0}^{log_c(n)} \left(\frac{b}{c}\right)^{i}$$

We have three interesting cases.

Case: b < c

We get: $T(n) \in O\left[(a-d)\, b^{log_c(n)} + d\, n \sum_{i=0}^{log_c(n)} \left(\frac{b}{c}\right)^{i}\right]$

$T(n) \in O[(a-d)\, b^{log_c(n)}] + O\left[d\, n \sum_{i=0}^{log_c(n)} \left(\frac{b}{c}\right)^{i}\right] = O[b^{log_c(n)}] + O(n)$

but we have $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{b^{log_c(n)}}{n} = \lim_{n\to\infty} \frac{b^{log_c(n)} ln(b)}{n\, ln(c)} =$

$= \lim_{n\to\infty} \frac{ln(b)}{ln(c)} \frac{b^{log_c(n)}}{n} = 0$

Hence, $T(n) \in O(n)$ since, $b^{log_c(n)}$ grows no faster than n.

Case: b = c

For $b = c$ we have

$$T(n) = (a-d)\, b^m + d\, n \sum_{i=0}^{log_c(n)} 1 = (a-d)n + d\, n\,(log_c(c) + 1)$$

and

$T(n) \in O[(a-d)\, n + d\, n\, (log_c(c)+1)] = O[n\, log_c(n)]$

Since, $\lim_{n \to \infty} \frac{n}{n \ ln(n)} = 0$

Case: b > c

We have

$$T(n) = (a - d)\ b^m + d\ n \sum_{i=0}^{m} (\frac{b}{c})^{i} = (a - d)\ b^m + d\ c^m \frac{1 - (\frac{b}{c})^{m+1}}{1 - \frac{b}{c}}$$

$$T(n) \in O\left[(a - d)\ b^m + d\ b^m \frac{1 - (\frac{c}{b})^{m+1}}{1 - \frac{c}{b}}\right] = O[b^m]$$

Finally we get: $T(n) \in O[b^{log_c(n)}] = O[n^{log_c(b)}]$