TSTE18 Digital Arithmetic
Seminar 7

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Division and square-root

- Restoring division
- Non-restoring division
- SRT division
- Higher-radix division

- **Reciprocals**
- **Division by convergence**
- **Square-rooting**

Reciprocal

- The reciprocal of D is denoted $\rho = 1/D$, with an estimate denoted $\tilde{\rho}$
- The division iteration looks like

$$r_i = br_{i-1} - z_i D$$

- Ideally, $r_i = 0$ leading to $br_{i-1} = z_i D$
- Therefore, the quotient selection in radix-b division can be approximately written as

$$z_i = \left\lceil \frac{br_{i-1}}{D} \right\rfloor \quad (1)$$

where the exact formulation depends on the division scheme

Reciprocal

- Recall from the SRT scheme that there is often an arbitrary choice of exact quotient digit
- We can now get

$$z_i = \left\lceil \frac{br_{i-1}}{D} \right\rfloor = \lceil br_{i-1}\rho \rfloor \approx \lceil br_{i-1}\tilde{\rho} \rfloor \quad (2)$$

- Hence, it seems like it may be possible to use an approximation of ρ to compute the next quotient digit
- Given a reciprocal, the digit can be selected based on a multiplication, attractive for high radix

High-radix division using approximated reciprocals

- Consider division of the dividend $X = 12\ 03\ 13 = r_0$ with the divisor $D = 24\ 17\ 22$ using radix-32
- The normalized approximated reciprocal is $\rho = \frac{1}{D} = \frac{01\ 00\ 00\ 00\ 00}{24\ 17\ 22} \approx 01\ 10 = \tilde{\rho}$
- Proceed according to

Iteration	$\tilde{\rho} r_{i-1}$	z_i	$z_i D$	$r_i = b r_{i-1} - z_i D$
1	15 28 15 02	15	11 16 9 10	00 19 03 0
2	25 02 26 28	25	19 05 26 06	−00 02 04 06
3	−02 25 15 28	−02	−01 17 03 12	00 19 02 20
...				

- Hence, the quotient is $\approx .\ 15\ 25\ \ -02 = .\ 15\ 24\ 30$ and the remainder $19\ 02\ 20 \times 32^{-3} = .19\ 02\ 20$
- In general, it is possible to compute a p by p digit reciprocal with p digit quotient and p digit remainder in $2\lceil p/k \rceil$ multiplications using radix b^k

High-radix division using approximated reciprocals

- A slightly better algorithm is obtained if we normalize the dividend and divisor as follows
- Consider the previous example $X = 12\ 03\ 13$, $D = 24\ 17\ 22$, and $\tilde{\rho} = 01\ 10$
- Normalize X and D with $\tilde{\rho}$ giving

$$X\tilde{\rho} = 15\ 28\ 15\ 02 = r_0 \quad (3)$$

$$D\tilde{\rho} = 01\ 00\ 07\ 06\ 28 = 01\ 00\ 00\ 00\ 00 + t \quad (4)$$

- The error in the normalized divisor is $t = 07\ 06\ 28$
- Make digit selection based on remainder and concatenate $z_i t$ as

Iteration	z_i	$z_i t$	$z_i\ z_i t$	r_i
1	15	03 12 07 04	15 03 12 07 04	25 02 26 28
2	25	05 20 11 28	25 05 20 11 28	−02 25 15 28
3	−02	...		

- Only one multiplication per iteration required leading to $\lceil p/k \rceil + 2$ in total, but the remainder is normalized with $\tilde{\rho}$

Newton-Raphson division

- Consider a reciprocal approximation $\tilde{\rho}_0 = \frac{1}{D}(1 - \epsilon)$ where $\epsilon < 1$ is the relative error
- If we can determine ϵ we can multiply with $1 + \epsilon$ and get $\tilde{\rho}_1 = \tilde{\rho}_0(1 + \epsilon) = \frac{1}{D}(1 - \epsilon^2)$ which is a better approximation
- Note that $1 - \epsilon = \tilde{\rho}_0 D \Rightarrow \epsilon = 1 - \tilde{\rho}_0 D$
- Hence, we want to multiply with $1 + \epsilon = 2 - \tilde{\rho}_0 D$
- This gives a recurrence

$$\tilde{\rho}_i = \tilde{\rho}_{i-1}(2 - \tilde{\rho}_{i-1} D) \quad (5)$$

- This type of recurrence is known as Newton-Raphson

Newton-Raphson reciprocal

- Example: Compute the reciprocal of $D = 0.75$ using an initial reciprocal estimation $\tilde{\rho}_0 = 1$

i	$1 - \tilde{\rho}_{i-1} D$	$2 - \tilde{\rho}_{i-1} D$	$\tilde{\rho}_i$	Rel. error to $1/D$
1	0.25	1.25	1.25	6.25×10^{-2}
2	6.25×10^{-2}	1.0625	1.328125	3.9062×10^{-3}
3	3.9062×10^{-3}	1.0039062	1.333313	1.5258×10^{-5}
4	1.5258×10^{-5}	1.000015258	1.3333333330	2.3283×10^{-10}
...				

Newton-Raphson division

- For a normalized divisor $\frac{1}{b} \leq D < 1$ the initial reciprocal estimation should be $1 \leq \tilde{\rho}_0 \leq \min\{b, \frac{3}{2}\frac{1}{D}\}$
- The reciprocal estimate after k iterations is

$$\tilde{\rho}_k = \tilde{\rho}_0 \prod_{j=0}^{k-1} \left(1 + \epsilon^{2^j}\right) = \frac{1}{D}\left(1 - \epsilon^{2^k}\right) \tag{6}$$

where ϵ is the error of the initial reciprocal approximation

- The Newton-Raphson reciprocal computation require $2k$ sequential multiplications
- For a division another multiplication is required to multiply the dividend with the reciprocal

Faster Newton-Raphson reciprocals

- A slight modification of the recurrence equation leads to half the number of sequential multiplications by allowing two to be performed in parallel

$$\tilde{\rho}_i = \tilde{\rho}_{i-1}(2 - D\tilde{\rho}_{i-1}) \tag{7}$$

$$D\tilde{\rho}_i = D\tilde{\rho}_{i-1}(2 - D\tilde{\rho}_{i-1}) \tag{8}$$

- Same example:

i	$2 - \tilde{\rho}_{i-1}D$	$\tilde{\rho}_i$	$D\tilde{\rho}_i$
1	1.25	1.25	0.9375
2	1.0625	1.328125	0.99609
3	1.0039062	1.333313	0.999985
...			

- Conceptually

$$\frac{1}{D} \approx \frac{1 \times \tilde{\rho}_0}{D \times \tilde{\rho}_0} \approx \frac{\tilde{\rho}_0(2 - D\tilde{\rho}_0)}{D\tilde{\rho}_0(2 - D\tilde{\rho}_0)} \approx \frac{\tilde{\rho}_1(2 - D\tilde{\rho}_1)}{D\tilde{\rho}_1(2 - D\tilde{\rho}_1)} \approx \cdots \approx \frac{\tilde{\rho}_k}{D\tilde{\rho}_k}$$

- Note that initializing the numerator to X instead of 1 will give X/D

Convergence division (Goldschmidt)

- The fact that we can initalize the numerator to X leads to the Goldschmidt convergence division algorithm
- Let an initial quotient approximation be $Q_0 = X\tilde{\rho}$ where $\tilde{\rho}$ is an initial reciprocal estimation
- Concurrently scale the divisor leading to a relative error $1 - \epsilon = D\tilde{\rho}$
- The result is then

$$\frac{X}{D} = \frac{X\tilde{\rho}}{D\tilde{\rho}} = \frac{Q_0}{1 - \epsilon} \tag{9}$$

- In the next iteration multiply with $1 + \epsilon$ to obtain

$$\frac{X}{D} = \frac{Q_0(1 + \epsilon)}{(1 - \epsilon)(1 + \epsilon)} = \frac{Q_1}{1 - \epsilon^2} \tag{10}$$

- Then

$$\frac{X}{D} = \frac{Q_1(1 + \epsilon^2)}{(1 - \epsilon^2)(1 + \epsilon^2)} = \frac{Q_2}{1 - \epsilon^4} \tag{11}$$

and so on

Convergence division (Goldschmidt)

- After k iterations $Q_k = \frac{X}{D}\left(1 - \epsilon^{2^k}\right)$ is obtained
- This gives that we have a good estimate of the number of iterations required to obtain a certain precision (assuming no rounding errors etc) given the reciprocal look-up table precision ϵ
- More iterations leads to more rounding errors as well, so in most implementations a reciprocal precision leading to two or three iterations are preferred

Convergence division (Goldschmidt)

- Example: Divide $X = 234$ with $D = 0.753$ obtaining a relative error smaller than 0.001 using two iterations
- With $\epsilon^{2^2} \leq 0.001$ we get $\epsilon \leq \sqrt[4]{0.001} \approx 0.17$
- In such a table we could find $\tilde{\rho} = 1.3$ leading to $Q_0 = 304.2$
- In parallel it is possible to compute $1 + \epsilon = D\tilde{\rho} = 0.9789$
- In the first iteration we obtain $1 - \epsilon = 2 - (1 + \epsilon)$ leading to $Q_1 = Q_0(1 - \epsilon) = 310.619$ and $1 - \epsilon^2 = (1 + \epsilon)(1 - \epsilon) = 0.999555$
- In the second iteration $1 + \epsilon^2 = 2 - \left(1 - \epsilon^2\right)$ leading to $Q_2 = Q_1\left(1 - \epsilon^2\right) = 310.7569$ and no need to compute $1 - \epsilon^4$ explicitly

Square root

- Computing the square root is in some ways a similar operation to division
- In the square root case we have $X = Z^2 + R$ where X is the radicand, Z is the root, and R is the remainder
- Compare with division: $X = DQ + R$
- Hence, we would like to divide with the resulting quotient

Square root

- Similarly we can iterate as, assuming $0 \leq X < 1$,

$$r_i = 2r_{i-1} - z_i(2Z_{i-1} + z_i 2^{-i}) \qquad (12)$$

  where $Z_i = \sum_{k=1}^{i} z_k 2^{-k}$ is the partially computed square root with $Z_0 = 0$ and $r_0 = X$
- Hence, we "divide" with the square root and update the square root estimate depending on if the estimate is too small/large
- Different selection rules for z_i is available corresponding to restoring, non-restoring and SRT division

Square root

- Consider the non-restoring square root with the selection rules

$$z_i = \begin{cases} 1, & r_{i-1} > 0 \\ -1, & r_{i-1} < 0 \end{cases} \qquad (13)$$

- If $r_{i-1} = 0$ the remainder is 0 and one can stop the recurrence
- This selection rule implies that if the remainder is positive, the current estimate of the square root is to small and vice versa

Square root example

- Consider computing the square root of $X = 0.1101011 = r_0$

i	z_i	r_i	Z_i
1	1	$1.1010110 - 0.1 = 1.0010110$	0.1
2	1	$10.0101100 - (1.0 + 0.01) = 1.000110$	0.11
3	1	$10.0011000 - (1.1 + 0.001) = 0.100010$	0.111
4	1	$1.0001000 - (1.11 + 0.0001) = -0.110000$	0.1111
5	-1	$-1.1000000 + (1.111 - 0.00001) = 0.010110$	$0.1111\bar{1}$
6	1	$0.101100 - (1.111\bar{1} + 0.000001) = -1.000111$	$0.1111\bar{1}1$
7	-1	$-10.001110 + (1.111\bar{1}1 - 0.0000001) = -1.\ldots$	$0.1111\bar{1}1\bar{1}$

- The result is $Z \approx 0.1111\bar{1}1\bar{1} = 0.1110101 = \tilde{Z}$
- $\tilde{Z}^2 = 0.11010101111001$

SRT square root

- An SRT scheme can be derived by normalizing the radicand to $1/4 \le X < 1$ leading to $1/2 \le Z < 1$
- The selection rule is

$$z_i = \begin{cases} 1, & 2r_{i-1} \ge 1/2 \quad \Leftrightarrow \quad 2r_{i-1} = 0X.Y\ldots, X+Y \ne 0 \\ 0, & -1/2 \le 2r_{i-1} < 1/2 \quad \Leftrightarrow \quad 2r_{i-1} = \begin{cases} 00.0\ldots \\ 11.1\ldots \end{cases} \\ -1, & 2r_{i-1} < -1/2 \quad \Leftrightarrow \quad 2r_{i-1} = 1X.Y, X+Y \ne 0\ldots \end{cases} \tag{14}$$

- Note that with this scheme and normalization $z_1 = 1$ and $r_1 = 2X - \frac{1}{2}$
- Problem: compute the square root of $X = 0.011101$ using the SRT scheme

Faster square roots

- It is also possible to derive square root algorithms corresponding to the earlier discussed algorithms reciprocal based division
- In particular we will look at the Newton-Raphson and Goldschmidt like convergence square roots
- First, define

$$Z_i = \sqrt{X}(1+\epsilon_i) \tag{15}$$

$$\rho_i = \frac{1}{\sqrt{X}}(1+\delta_i) \tag{16}$$

as the ith iterative square root and square root reciprocal, respectively

- Note that $\frac{Z_i}{\rho_i} = \frac{\sqrt{X}(1+\epsilon_i)}{\frac{1}{\sqrt{X}}(1+\delta_i)} = X\frac{1+\epsilon_i}{1+\delta_i}$

Newton-Raphson square root

- The Newton-Raphson square root recurrence is

$$q_i = \frac{1}{2}\left(q_{i-1} + \frac{X}{q_{i-1}}\right) \tag{17}$$

- Hence, a reciprocal/division is required in each iteration
- This can be alleviated by computing the square root reciprocal leading to a recurrence

$$\rho_i = \rho_{i-1}\left(\frac{1}{2}\left(3 - \rho_{i-1}^2 X\right)\right) \tag{18}$$

- This leads to a single reciprocal after the final iteration

Newton-Raphson square root

- The Newton-Raphson square root approach can be rewritten such that an approximation of the root reciprocal is used in the square root recurrence as

$$
\begin{aligned}
q_i &= q_{i-1} + \frac{\rho}{2}\left(x - q_{i-1}^2\right) && (19)\\
\rho_i &= 2\rho_{i-1} - \rho_{i-1}^2 q_i && (20)
\end{aligned}
$$

Goldschmidt convergence square root

- The corresponding recurrence equations for convergence square root is

$$
\begin{aligned}
q_i &= q_{i-1}\left(\frac{1}{2}\left(3 - \rho_{i-1}q_{i-1}\right)\right) && (21)\\
\rho_i &= \rho_{i-1}\left(\frac{1}{2}\left(3 - \rho_{i-1}q_{i-1}\right)\right) && (22)
\end{aligned}
$$

- The initial root reciprocal is obtained from a table as $\rho_0 = \frac{1}{\sqrt{X}}(1+\epsilon_0)$ and the compute the initial root as $q_0 = x\rho_0 = \sqrt{X}(1+\epsilon_1)$
- In this way, both approximations have the same relative error and $\frac{q_i}{\rho_i} = X$
- The ith complementary error factor is then $1 - \epsilon_{i-1} - \frac{1}{2}\epsilon_{i-1}^2 = \frac{1}{2}(3 - q_{i-1}\rho_{i-1})$