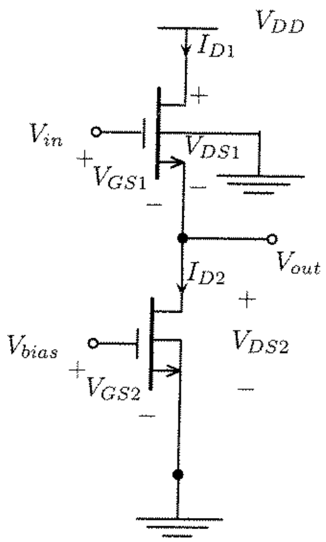
SOLUTIONS. Exam March 13, 2008

TSTE08 Analog and Discrete-time Integrated Circuits.

Excercise 1.

For transistor M1 we have

$$\left(\frac{W}{L}\right)_1 = \frac{I_{D1}}{\frac{1}{2}\mu_0 C_{ox}(V_{GS1} - V_{Tn})^2(1 + \lambda(V_{DS1} - V_{eff1}))} \quad (1)$$



- $I_{D1} = I_D = 20 \cdot 10^{-9}$ A from text.
- $\mu_0 C_{ox} = 20 \cdot 10^{-9}$ A/V^2 from text.
- $V_{GS1} = V_{in} - V_{out} = 1.5$ V .
- $V_{Tn} = V_{T,0} + \gamma(\sqrt{2\phi_F - V_{BS}} - \sqrt{2\phi_F})$ where $V_{BS} = 0 - V_{out} = -1.5$ V
  i.e. $V_{Tn} = 0.5 + 0.6(\sqrt{2 \cdot 0.4 + 1.5} - \sqrt{2 \cdot 0.4}) \approx 0.8733$ V
- $V_{DS1} = V_{DD} - V_{out,DC} = 1.5$ V from text.
- $V_{eff1} = V_{GS1} - V_{Tn} \approx 1.5 - 0.8733 = 0.6267$ V.

Inserting in eqn.(1) gives:

$$\left(\frac{W}{L}\right)_1 = \frac{20 \cdot 10^{-9}}{0.5 \cdot 20 \cdot 10^{-9} \cdot 0.6267^2 \cdot (1 + 0.03(1.5 - 0.6267))} \approx 4.96 \quad (2)$$

For transistor M2 we have

$$\left(\frac{W}{L}\right)_2 = \frac{I_{D2}}{\frac{1}{2}\mu_0 C_{ox}(V_{GS2} - V_{Tn})^2(1 + \lambda(V_{DS2} - V_{eff2}))} \quad (3)$$

- $I_{D2} = I_D = 20 \cdot 10^{-9}$ A from text.
- $\mu_0 C_{ox} = 20 \cdot 10^{-9}$ A/V^2 from text.

- $V_{GS2} = V_{bias} = 1$ V.
- $V_{Tn} = 0.5$ V ($V_{BS2} = 0$)
- $V_{DS2} = V_{out,DC} = 1.5$ V from text.
- $V_{eff1} = V_{GS2} - V_{Tn} = 0.5$ V.

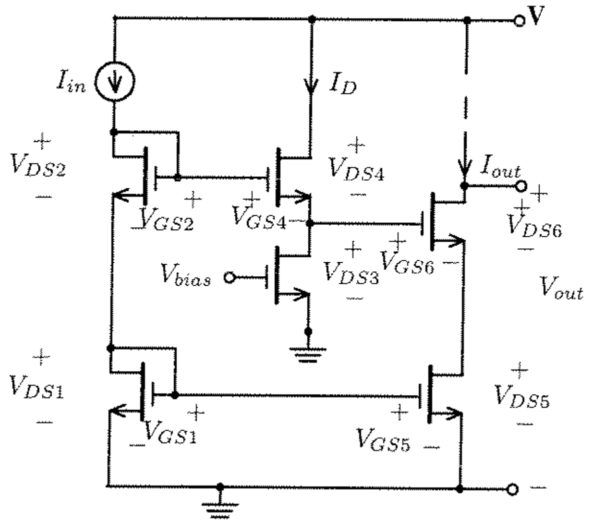
Inserting in eqn. (3) gives:

$$\left(\frac{W}{L}\right)_2 = \frac{20 \cdot 10^{-9}}{0.5 \cdot 20 \cdot 10^{-9} \cdot 0.5^2 \cdot (1 + 0.03(1.5 - 0.5))} \approx 7.77 \tag{4}$$

Answer: $\left(\frac{W}{L}\right)_1 \approx 4.96$ och $\left(\frac{W}{L}\right)_2 \approx 7.77$

Exercise 2.

The minimum output voltage is found by starting at the ground node and finding all paths to the output node. Here we have three different ways, as we can start either at **M1**, **M3** or **M5**.



As $V_{DS1} = V_{GS1}$ and $V_{eff1} = V_{GS1} - V_{t1}$ the formula $I_D = \alpha(V_{GS} - V_t)^2(1 + \lambda(V_{DS} - V_{eff}))$ gives following expression for the current through **M1**, i.e., I_{in}:

$$I_{in} = \alpha_1(V_{GS1} - V_{t1})^2(1 + \lambda(V_{GS1} - (V_{GS1} - V_{t1}))) = \alpha_1(V_{GS1} - V_{t1})^2(1 + \lambda V_{t1}) \tag{5}$$

which gives the relation

$$V_{GS1} = \sqrt{\frac{I_{in}}{\alpha_1(1 + \lambda V_{t1})}} + V_{t1} \tag{6}$$

In the same way we also can show that

$$V_{GS2} = \sqrt{\frac{I_{in}}{\alpha_2(1 + \lambda V_{t2})}} + V_{t2} \tag{7}$$

We also note that **M3** and **M4** are at the limit of saturation, which means that $V_{DS3} = V_{eff3}$ and $V_{DS4} = V_{eff4}$ and then

$$I_D = \alpha_3(V_{GS3} - V_{t3}) \Rightarrow V_{DS3} = V_{GS3} - V_{t3} = \sqrt{\frac{I_D}{\alpha_3}} \tag{8}$$

and

$$I_D = \alpha_4(V_{GS4} - V_{t4}) \Rightarrow V_{GS4} = \sqrt{\frac{I_D}{\alpha_4}} + V_{t4} \tag{9}$$

Also note that the minimum values of V_{DS5} and V_{DS6} are V_{eff4} and V_{eff5}, respectively. That gives

$$V_{DS5min} = V_{GS5} - V_{t5} = \sqrt{\frac{I_{out}}{\alpha_5}} \tag{10}$$

and

$$V_{DS6min} = V_{GS6} - V_{t6} = \sqrt{\frac{I_{out}}{\alpha_6}} \tag{11}$$

Eqn. (11) gives

$$V_{GS6} = V_{DS6min} + V_{t6} \tag{12}$$

- Starting at **M1** gives:
  $V_{out} = V_{GS1} + V_{GS2} - V_{GS4} - V_{GS6} + V_{DS6} \Rightarrow V_{outmin} = V_{GS1} + V_{GS2} - V_{GS4} - (V_{DS6min} + V_{t6}) + V_{DS6min} = V_{GS1} + V_{GS2} - V_{GS4} - V_{t6}$
  I.e. In this case $V_{out,min}$ will be

$$V_{outmin} = V_{GS1} + V_{GS2} - V_{GS4} - V_{t6} \tag{13}$$

- Starting at **M3** gives:

$$V_{outmin} = V_{DS3min} - (V_{DS6min} + V_{t6}) + V_{DS6min} = V_{DS3min} - V_{t6} \tag{14}$$

- Starting at **M5** gives:

$$V_{outmin} = V_{DS5min} + V_{DS6min} \tag{15}$$

Answer:
Introduceing eqns. (4)-(12) in eqns. (13)-(15) gives that $V_{out,min}$ will be the **maximum value** from following three expressions (16), (17) or (18):

$$\sqrt{\frac{I_{in}}{\alpha_1(1+\lambda V_{t1})}} + V_{t1} + \sqrt{\frac{I_{in}}{\alpha_2(1+\lambda V_{t2})}} + V_{t2} - \sqrt{\frac{I_D}{\alpha_4}} - V_{t4} - V_{t6} \tag{16}$$

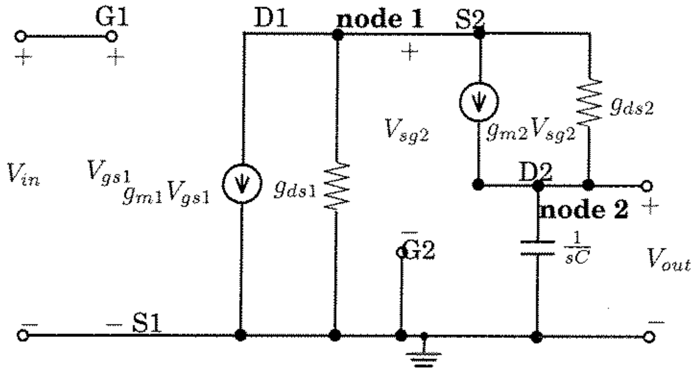
$$\sqrt{\frac{I_D}{\alpha_3}} - V_{t6} \tag{17}$$

$$\sqrt{\frac{I_{out}}{\alpha_5}} + \sqrt{\frac{I_{out}}{\alpha_6}} \tag{18}$$

Which value will be the largest one depends on the values of the currents and the constants.

Exercise 3.

a) Small signal equivalent circuit: (Note that $V_{gs1} = V_{in}$)



b)

$$\begin{cases} \underline{\text{Nod 1}}: g_{m1}V_{in} + g_{ds1}V_{sg2} = -(g_{m2}V_{sg2} + gds2(V_{sg2} - V_{out})) \\ \underline{\text{Nod 2}}: g_{m2}V_{sg2} + g_{ds2}(V_{sg2} - V_{out}) = sCV_{out} \end{cases} \tag{19}$$

$$(19) \Rightarrow g_{m1}V_{in} + g_{ds1}V_{sg2} = -sCV_{out} \tag{20}$$

$$(19) \Rightarrow V_{sg2}(g_{m2} + g_{ds2}) = V_{out}(sC + g_{ds2}) \Rightarrow V_{sg2} = \frac{sC + g_{ds2}}{g_{m2} + g_{ds2}}V_{out} \tag{21}$$

$$(20), (21) \Rightarrow g_{m1}V_{in} + g_{ds1} \cdot \frac{sC + g_{ds2}}{g_{m2} + g_{ds2}}V_{out} = -sCV_{out} \tag{22}$$

$$(22) \Rightarrow \underline{\underline{H(s)}} = \frac{V_{out}}{V_{in}} = \frac{-g_{m1}}{sC + \frac{g_{ds1}(sC+g_{ds2})}{g_{m2}+g_{ds2}}} = \underline{\underline{\frac{-g_{m1}}{\frac{g_{ds1}g_{ds2}}{g_{m2}+g_{ds2}} + \left(1 + \frac{g_{ds1}}{g_{m2}+g_{ds2}}\right)sC}}} \tag{23}$$

c)

$$g_{m2} >> g_{ds2} \Rightarrow \underline{\underline{H(s)}} \approx \frac{-g_{m1}}{\frac{g_{ds1}g_{ds2}}{g_{m2}} + \left(1 + \frac{g_{ds1}}{g_{m2}}\right)sC} = \underline{\underline{-\frac{g_{m1}g_{m2}}{g_{ds1}g_{ds2}} \cdot \frac{1}{1 + \left(\frac{g_{m2}}{g_{ds1}g_{ds2}} + \frac{1}{g_{ds2}}\right)sC}}} \tag{24}$$

$$\Rightarrow H(\omega) = -\frac{g_{m1}g_{m2}}{g_{ds1}g_{ds2}} \cdot \frac{1}{1 + j\omega C\left(\frac{g_{m2}}{g_{ds1}g_{ds2}} + \frac{1}{g_{ds2}}\right)} \Rightarrow |H(\omega)| = \frac{g_{m1}g_{m2}}{g_{ds1}g_{ds2}} \cdot \frac{1}{\sqrt{1 + \omega^2C^2\left(\frac{g_{m2}}{g_{ds1}g_{ds2}} + \frac{1}{g_{ds2}}\right)^2}} \tag{25}$$

$\omega = 0$ gives the DC-gain $A_0 = \underline{\underline{\frac{g_{m1}g_{m2}}{g_{ds1}g_{ds2}}}}$

$|H(\omega)| = 1$ gives the unity-gain-frequency. I.e.

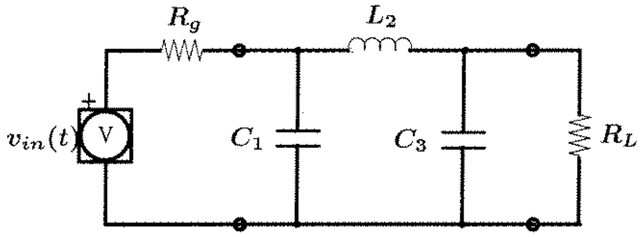
$$\sqrt{1 + \omega^2C^2\left(\frac{g_{m2}}{g_{ds1}g_{ds2}} + \frac{1}{g_{ds2}}\right)^2} = \frac{g_{m1}g_{m2}}{g_{ds1}g_{ds2}} \Rightarrow \omega_u = \frac{\left(\left(\frac{g_{m1}g_{m2}}{g_{ds1}g_{ds2}}\right)^2 - 1\right)^{1/2}}{C\left(\frac{g_{m2}}{g_{ds1}g_{ds2}} + \frac{1}{g_{ds2}}\right)} \tag{26}$$

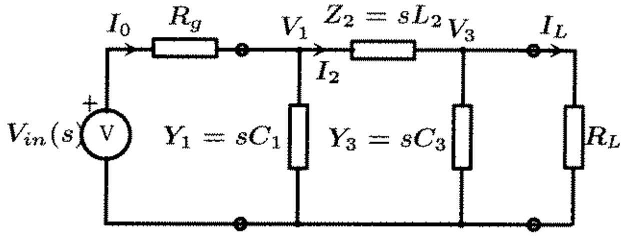
As $g_{m1} >> g_{ds1}$ and $g_{m2} >> g_{ds2}$ we have following approximation

$$\underline{\underline{\omega_u}} \approx \frac{\frac{g_{m1}g_{m2}}{g_{ds1}g_{ds2}}}{C \cdot \frac{g_{m2}}{g_{ds1}g_{ds2}}} = \underline{\underline{\frac{g_{m1}}{C}}} \tag{27}$$

Exercise 4.
See figures below!

Ladder-filter. 3:d order.



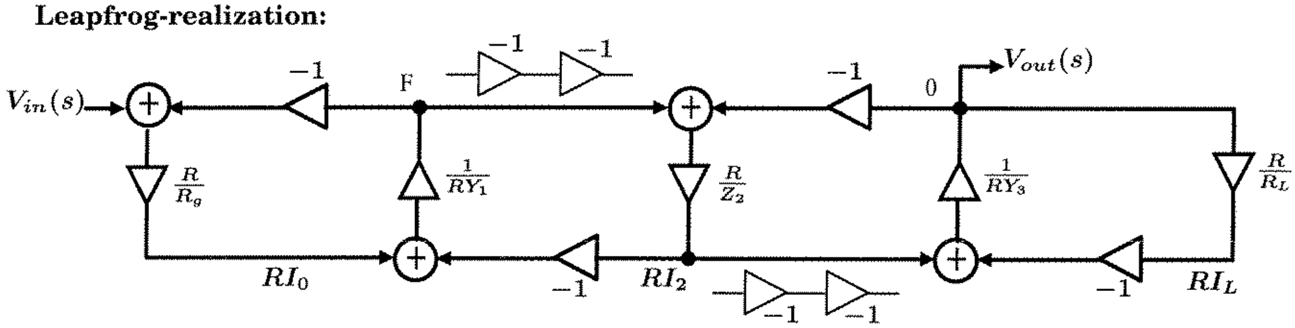


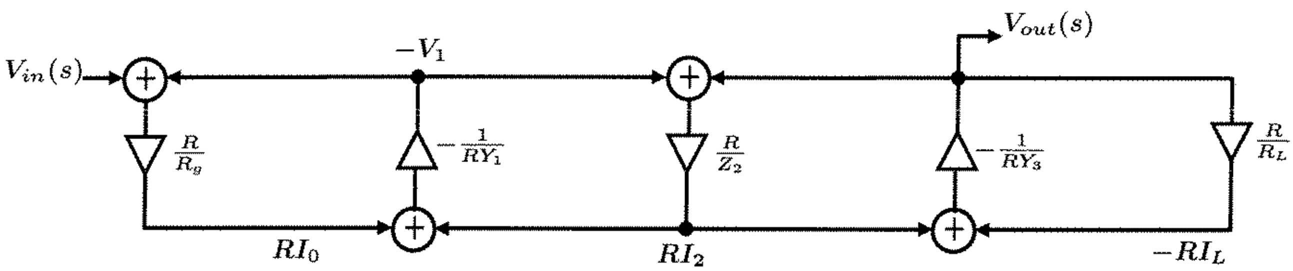
Express the currents in voltages and voltages in currents. Then introduce the constant R to achieve all quantities to be voltages.

$$
\begin{aligned}
I_0 &= \frac{V_{in} - V_1}{R_g} & V_1 &= \frac{I_0 - I_2}{Y_1} & RI_0 &= \frac{R}{R_g}(V_{in} - V_1) & V_1 &= \frac{RI_0 - RI_2}{RY_1} \\
I_2 &= \frac{V_1 - V_3}{Z_2} & V_3 &= \frac{I_2 - I_L}{Y_3} & RI_2 &= \frac{R}{Z_2}(V_1 - V_3) & V_3 &= \frac{RI_2 - RI_L}{RY_3} \\
I_L &= \frac{V_3}{R_L} & & & I_L &= \frac{R}{R_L}V_3
\end{aligned}
\tag{28}
$$

These equations give following leap-frog realization. -1 propagation gives the second figure below. The first figure on next page is a modified leap-frog realization, where one adder has been eliminated.

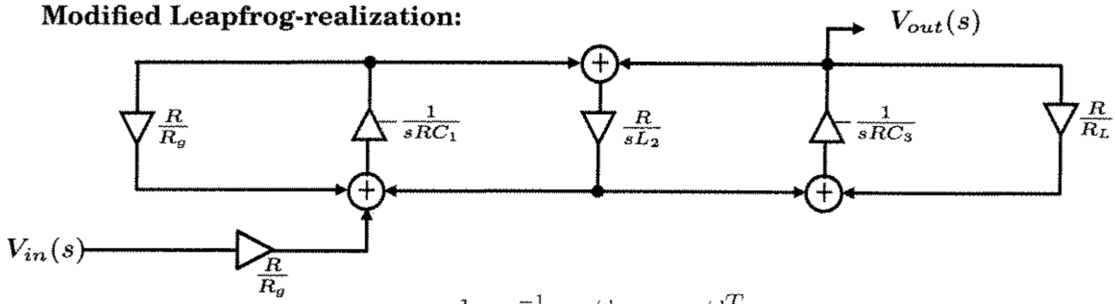
Leapfrog-realization:



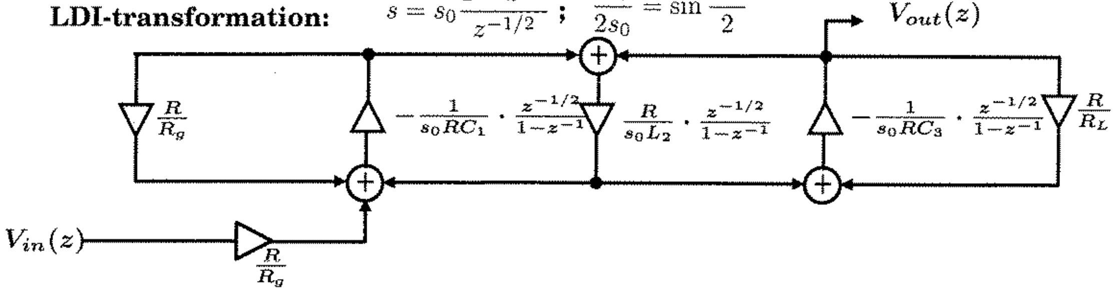


LDI-transformation of a modified leap-frog filter, followed by $z^{-1/2}$ propagation":

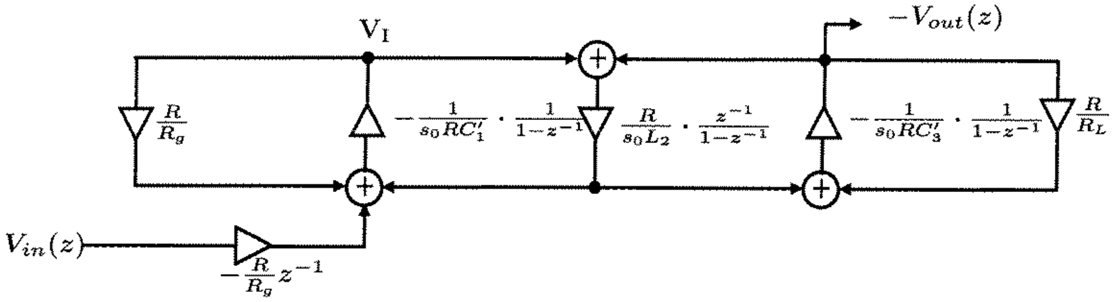
Modified Leapfrog-realization:



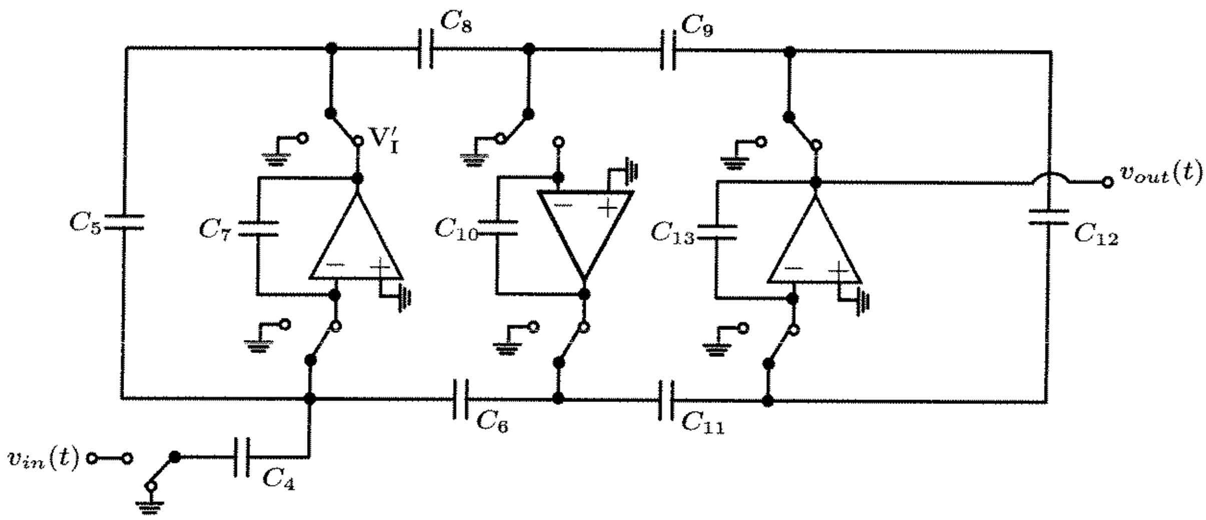
LDI-transformation: $s = s_0 \frac{1 - z^{-1}}{z^{-1/2}}$; $\frac{\omega_a}{2 s_0} = \sin \frac{\omega T}{2}$



$z^{-1/2}$**-propagation and corrections:**



SC-filter-implementation:



Comments: "$z^{-1/2}$ propagation" actually gives $\frac{R}{R_g}z^{-1/2}$ and $\frac{R}{R_L}z^{-1/2}$, respectively. Deleting $z^{-1/2}$ is equivalent to replace R_g with $R_g z^{-1/2} \to R_g e^{-j\omega T/2} = R_g(\cos\frac{\omega T}{2} - j\sin\frac{\omega T}{2})$

I.e. $\frac{1}{\frac{1}{Rgz^{-1/2}}} = \frac{1}{\frac{z^{+1/2}}{Rg}} \Rightarrow \frac{1}{\frac{1}{R_g(\cos\frac{\omega T}{2} - j\sin\frac{\omega T}{2})}} = \frac{1}{\frac{\cos\frac{\omega T}{2}}{R_g} + j\frac{\sin\frac{\omega T}{2}}{R_g}}$

As $\sin\frac{\omega T}{2} = \frac{\omega_a}{2s_0}$ when $\cos\frac{\omega T}{2} = \sqrt{1 - \left(\frac{\omega_a}{2s_0}\right)^2}$ this means that deleting $z^{-1/2}$ is the same thing as to replace $\frac{1}{\frac{1}{R_g}} = R_g$ with $\frac{1}{\sqrt{1-\left(\frac{\omega_a}{2s_0}\right)^2}\cdot\frac{1}{R_g} + j\frac{\omega_a}{2s_0 R_g}}$

I.e. R_g will be changed to a resistor $\frac{R_g}{\sqrt{1-\left(\frac{\omega_a}{2s_0}\right)^2}}$ paralell to a capcitor $\frac{1}{2s_0 R_g}$. To compensate for that we should: (Note that the voltage source in series with R_g can be replaced by its Norton equivalent, i.e. with a current source paralell to R_g which simplifies this interpretation.)

1) Replace R_g with a frequency dependent resistor $R_g\sqrt{1-\left(\frac{\omega_a}{2s_0}\right)^2} = R_g\cos\frac{\omega T}{2}$, but this compensation i very small unless ωT is not too large (i.e if $f_s >> 2f_{cutoff}$), so we neglect this compensation at least for filters with small bandwidth.

2) Replace capacitor C_1 with $C_1' = C_1 - \frac{1}{2s_0 R_g}$, which has been done in the figure above. In the same way C_3 has been replaced by $C_3' = C_3 - \frac{1}{2s_0 R_L}$

"$z^{-1/2}$ propagation" also gives $\frac{R}{R_g}z^{-1/2}$ on the input, which has been replaced by $-\frac{R}{R_g}z^{-1}$. The change to z^{-1} yields a time delay, but as the input is a time-continuos signal that is of no significance. The change of sign to "minus" ($-$) means a phase shift of 180^o, i.e. the output will change sign, but gives us the opportunity to use a non-inverting accumulator on the input of the SC-realization. Which means that the input automatically will be sampled, and we need no extra sample-and-hold circuit at the input.

Coefficient identification.
Equations for V_I and V_I':

$$\begin{cases} V_I = \frac{1}{s_0 R_g C_1'}\cdot\frac{z^{-1}}{1-z^{-1}}\cdot V_{in} - \frac{1}{s_0 R_g C_1'}\cdot\frac{1}{1-z^{-1}}\cdot V_I - \frac{1}{s_0 R C_1'}\cdot\frac{1}{1-z^{-1}}\cdot V_{II} \\ V_I' = \frac{C_4}{C_7}\cdot\frac{z^{-1}}{1-z^{-1}}\cdot V_{in} - \frac{C_5}{C_7}\cdot\frac{1}{1-z^{-1}}\cdot V_I' - \frac{C_6}{C_7}\cdot\frac{1}{1-z^{-1}}\cdot V_{II}' \end{cases} \tag{29}$$

Identification gives:

$$\begin{cases} \frac{C_4}{C_7} = \frac{1}{s_0 R_g C_1'} = \frac{1}{s_0 R_g (C_1 - \frac{1}{2s_0 R_g})} = \frac{1}{s_0 R_g C_1 - 0.5} \\ \frac{C_5}{C_7} = \frac{1}{s_0 R_g C_1'} = \frac{1}{s_0 R_g C_1 - 0.5} \\ \frac{C_6}{C_7} = \frac{1}{s_0 R C_1'} = \frac{1}{s_0 R (C_1 - \frac{1}{2s_0 R_g})} \end{cases} \tag{30}$$

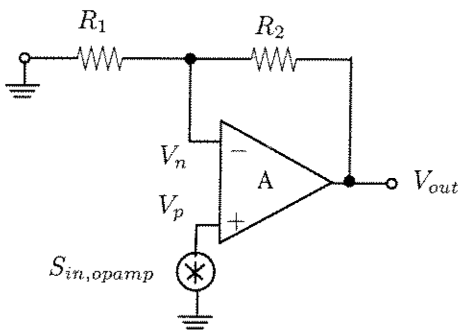
Exercise 5.

a) The output noise spectral density can be computed by the following formula

$$S_{out}(\omega) = |H(\omega)|^2 S_{in}(\omega) \tag{31}$$

where $H(\omega)$ is the transfer function from the noise source to the output node.

Determin the transfer function from the positive input node to the output of the operational amplifier, while the input voltage is zeroed:



Kirchhoff's current law gives:

$$\begin{aligned} (0 - V_n)G_1 &= (V_n - V_{out})G_2 \\ (V_p - V_n)A &= V_{out} \end{aligned} \tag{32}$$

Solving this system of equations results in

$$H = \frac{V_{out}}{V_p} = \frac{G_1 + G_2}{G_2 + (G_1 + G_2)/A} = \frac{(G_1 + G_2)g_{m1}}{G_2 g_{m1} + (G_1 + G_2)g_{out}} \tag{33}$$

Hence, the equivalent output noise spectral density is given by

$$S_{out} = |H|^2 \cdot S_{in,opamp} = \left(\frac{(G_1 + G_2)g_{m1}}{G_2 g_{m1} + (G_1 + G_2)g_{out}}\right)^2 \cdot S_{in,opamp} \tag{34}$$

Using the fact that the ratio between the resistors is equal to a gives the answer:

$$\underline{\underline{S_{out} = \frac{(1+a)^2 g_{m1}}{(g_{m1} + (1+a)g_{out})^2} \cdot \frac{16kT}{3}(1 + \frac{g_{m4}}{g_{m1}})}} \tag{35}$$

b) The noise at the output can be decreased by increasing the transconductance g_{m1} of the input stage. This decreases the last term in Equation (35) while the first part is not changed so much. This will increase the gain of the amplifier.