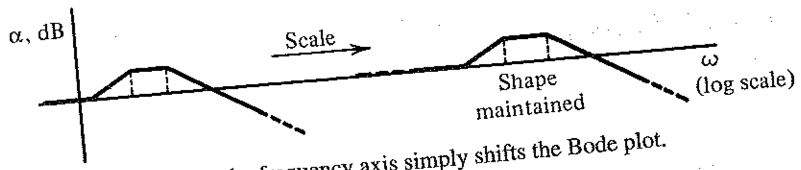


Figure 3.22 Scaling the frequency axis simply shifts the Bode plot.

3.4 ACTIVE REALIZATIONS

3.4.1 Inverting Opamp Circuits

We mentioned at the end of Section 3.2 that passive realizations of the function

$$T(s) = \frac{N(s)}{D(s)} = K\frac{s+z_1}{s+p_1} \tag{3.66}$$

have a number of problems that we proposed to solve with the aid of operational amplifiers. To determine the circuit that might provide a solution, we consult Chapter 2 and notice that the inverting amplifier may be suitable if we replace the resistors by impedances. Figure 3.23 shows the circuit. The analysis of this circuit is no different from its resistive version in Chapter 2. We sum the currents at the inverting input node of the opamp or, for simplicity, consult Eq. (2.55) and replace R_i by Z_i. The result is

$$\frac{V_2}{V_1} = -\frac{Z_2}{Z_1}\frac{1}{1+\dfrac{1}{A(s)}\left(1+\dfrac{Z_2}{Z_1}\right)} = -\frac{Y_1}{Y_2}\frac{1}{1+\dfrac{1}{A(s)}\left(1+\dfrac{Y_1}{Y_2}\right)} \tag{3.67a}$$

On the right-hand side we have used admittances, $Y = 1/Z$, because in the following, as is often the case in active filter work, the treatment will be simpler and more transparent if it is based on admittances. Also, we have included in Eq. (3.67a) the effect on the transfer function of finite and frequency-dependent opamp gain $A(s)$ to remind ourselves that this is a concern we must address. Initially, however, we shall assume $A = \infty$ so that we can focus on the main issues involved in designing first-order circuits. Thus we base our further treatment on

$$T(s) = \frac{V_2}{V_1} \approx -\frac{Z_2}{Z_1} = -\frac{Y_1}{Y_2} \tag{3.67b}$$

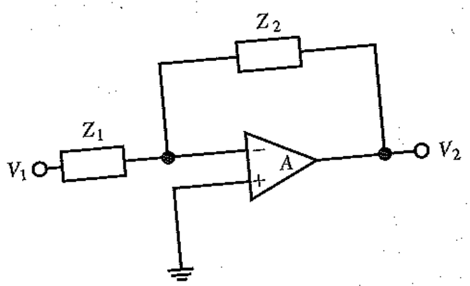


Figure 3.23 Inverting amplifier with impedances in the passive feedback network.

and contemplate how the bilinear function of Eq. (3.66), multiplied, of course, by −1, can be realized by the circuit in Fig. 3.23,

$$T(s) = -\frac{Z_2}{Z_1} = -\frac{Y_1}{Y_2} = -K\frac{s+z_1}{s+p_1} \tag{3.68}$$

The problem to be considered may be formulated in terms of this equation. We assume that the specifications of the design problem are the values K, z_1, and p_1. These may be found from a Bode plot—the break frequencies and the gain at some frequency—or obtained in any other convenient way. The solution to the design problem involves finding a circuit and the elements in that circuit. We will assume that inductors are excluded from our considerations. Hence we need to find the values of the Rs and Cs and their interconnections. Once the components are found, they are adjusted by appropriate frequency scaling and impedance-level scaling as discussed in Chapter 1, Section 4, to obtain convenient element values. Finally, the components may be tuned as necessary to realize the prescribed behavior exactly.

The procedure we will follow requires that some parts of the prescribed right-hand side of Eq. (3.68) be assigned to Y_1 and Y_2. The assignments are not unique, resulting in several different design strategies and circuits. Since inductors are excluded, we must avoid making the identifications $Y = 1/(sL)$ or $Z = sL$. One of several possibilities that suggests itself is to make the admittances linear functions of frequency,

$$Y_1 = sC_1 + G_1 \quad \text{and} \quad Y_2 = sC_2 + G_2 \tag{3.69}$$

to give

$$T(s) = -\frac{Y_1}{Y_2} = -\frac{sC_1+G_1}{sC_2+G_2} = -\frac{C_1}{C_2}\frac{s+G_1/C_1}{s+G_2/C_2} \tag{3.70}$$

from which we can identify

$$z_1 = \frac{G_1}{C_1} = \frac{1}{C_1R_1}, \quad p_1 = \frac{G_2}{C_2} = \frac{1}{C_2R_2}, \quad \text{and} \quad K = \frac{C_1}{C_2} \tag{3.71}$$

The circuit is shown in Fig. 3.24. With Eq. (3.71) it is easy to find the component values for a prescribed design.

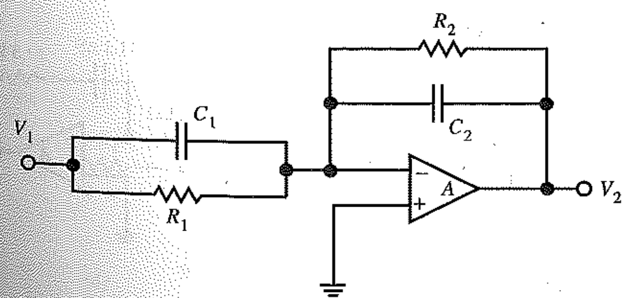


Figure 3.24 Active circuit realizing the bilinear function of Eq. (3.70).

EXAMPLE 3.5

Design a circuit to realize the bilinear transfer function with a zero at $f_z = 830$ Hz, a pole at $f_p = 13$ kHz, and a high-frequency gain of 23 dB.

Solution

On a linear scale, the high-frequency gain corresponds to

$$K = 10^{23/20} = 14.125$$

Thus, the transfer function of the circuit is

$$T(s) = -14.125 \frac{s + 2\pi \times 830 \text{ Hz}}{s + 2\pi \times 13{,}000 \text{ Hz}} \tag{3.72}$$

Although the element values could be calculated directly from this equation with the help of Eq. (3.71), let us gain some practice in normalizing equations. If the frequency parameter is scaled by $\omega_S = 2\pi \times 1000$ rad/s, the result is

$$T(s) = -14.125 \frac{s_n + 0.83}{s_n + 13} \tag{3.73}$$

where $s_n = s/\omega_S$. We have then, with normalized elements (identified by the subscript n),

$$G_{1n}/C_{1n} = 0.83, \quad G_{2n}/C_{2n} = 13, \quad \text{and} \quad C_{1n}/C_{2n} = 14.125 \tag{3.74}$$

Since we have four elements and only three constraints, we are free to choose one component arbitrarily. Let us select C_1 and determine the remaining elements as functions of C_1:

$$R_{1n} = \frac{1}{0.83 C_{1n}}, \quad C_{2n} = \frac{C_{1n}}{14.125}, \quad \text{and} \quad R_{2n} = \frac{1}{13 C_{2n}} = \frac{14.125}{13 C_{1n}}$$

Letting now $C_1 = 50$ nF $= 0.05\ \mu$F, and recalling that the frequency is normalized by $\omega_S = 2\pi \times 1000$ rad/s, results in

$$R_1 = \frac{1}{2\pi \times 830 \text{ Hz} \times 50 \text{ nF}} = 3.835 \text{ k}\Omega, \quad C_2 = \frac{C_1}{14.125} = 3.54 \text{ nF}$$

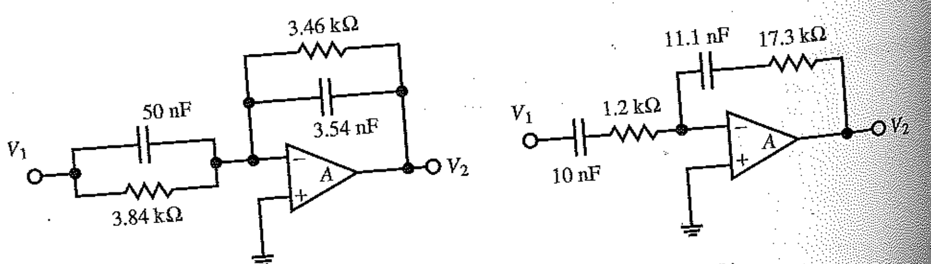


Figure 3.25 Circuit of (a) Example 3.5; (b) Example 3.6. Both circuits realize the same transfer function, Eq. (3.72).

and

$$R_2 = \frac{1}{2\pi \times 13{,}000 \text{ s}^{-1} \times C_2} = 3.458 \text{ k}\Omega$$

We see that at the origin $T(0) = -R_2/R_1 = -0.901$ and at high frequencies $T(\infty) = -C_1/C_2 = -14.125$. The circuit, an (inverting) highpass filter, is shown in Fig. 3.25a.

A comment is appropriate at this point concerning the problems we pointed out at the end of Section 3.2. Observe first that the filter in Fig. 3.23 can be loaded without affecting the transfer function in any substantial way because of the low (ideally zero) output impedance of the opamp. Second, as mentioned, it is clear from Eq. (3.70) that pole and zero can be adjusted independently. Since the zero is provided by $Y_1 = 1/Z_1$ and the pole is given by $Y_2 = 1/Z_2$, varying the zero by R_1 or C_1 will not affect the pole and, conversely, varying R_2 or C_2 will not change the zero. Finally, apart from being on the negative real axis, there are no restrictions on the pole or zero locations. For instance, we may choose and $C_1 = R_2 = 0$ to obtain the circuit in Fig. 3.26a. It realizes

$$T(s) = -\frac{Y_1}{Y_2} = -\frac{1}{sC_2R_1} \tag{3.75}$$

This is an inverting ideal integrator, a function that could not have been realized with only passive elements. Its attenuation can be written as

$$\alpha(\omega) = 20 \log |T(j\omega)| = -20 \log(\omega\tau) = -20 \log\left(\frac{\omega}{1/\tau}\right) \tag{3.76}$$

where $\tau = C_2R_1$ is the integrator time constant, and the phase is $\theta(\omega) = 180° - 90°$. In this equation, 180° arises from the minus sign in Eq. (3.75) and the integration itself, $1/(s\tau)$, contributes $-90°$. Figure 3.27a shows a Bode plot of the integrator magnitude with its breakpoint $1/\tau$. We will find integration, and therefore the ideal integrator, to be very important and fundamental in the design of active filters. A separate section, Section 4.4, will be devoted to its design.

A second function that cannot be implemented with only passive components is the differentiator with transfer function

$$T(s) = \frac{V_2}{V_1} = s\tau \tag{3.77}$$

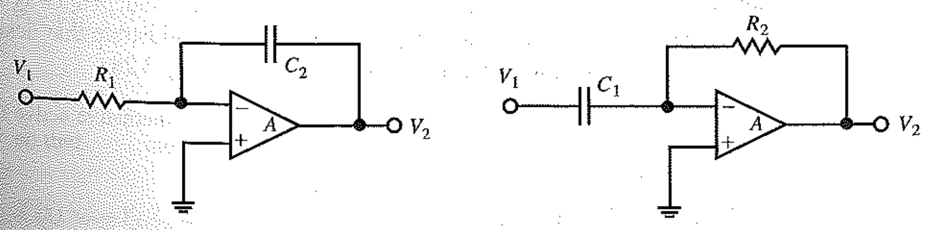


Figure 3.26 (a) Integrator; (b) differentiator.

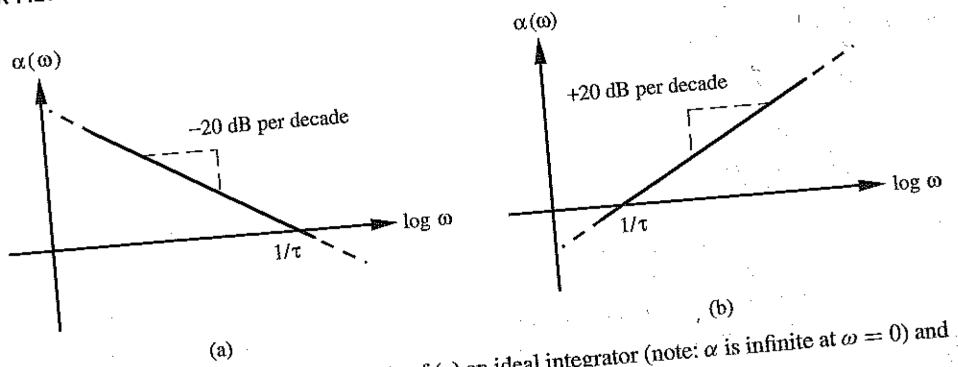


Figure 3.27 Bode plot of the magnitude of (a) an ideal integrator (note: α is infinite at $\omega = 0$) and (b) an ideal differentiator.

If we accept a minus sign for an inverting differentiator, the function is readily implemented by the circuit in Fig. 3.24 with $R_1 = C_2 = 0$ as shown in Fig. 3.26b. It realizes

$$T(s) = \frac{V_2}{V_1} = -\frac{sC_1}{G_2} = -sC_1R_2 = -s\tau \tag{3.78}$$

where $\tau = C_1R_2$ is the differentiator time constant and sets the breakpoint on the Bode plot. The Bode plot of the differentiator gain is shown in Fig. 3.27b.

We mentioned earlier that there are several possibilities for identifying the impedances in Fig. 3.23 such that the circuit realizes the bilinear transfer function of Eq. (3.66). In other words, circuit design does not normally lead to a unique solution. To demonstrate another general solution to our problem, let us identify instead of Eq. (3.69) the first-order impedances

$$Z_1(s) = R_1 + \frac{1}{sC_1} \quad \text{and} \quad Z_2(s) = R_2 + \frac{1}{sC_2} \tag{3.79}$$

to obtain the circuit in Fig. 3.28. Inserting these values into Eq. (3.68) results in

$$T(s) = -\frac{Z_2}{Z_1} = -\frac{R_2 + 1/(sC_2)}{R_1 + 1/(sC_1)} = -\frac{R_2}{R_1}\frac{s + 1/(C_2R_2)}{s + 1/(C_1R_1)} = -K\frac{s + z_1}{s + p_1} \tag{3.80}$$

where we have

$$K = \frac{R_2}{R_1}, \quad z_1 = \frac{1}{C_2R_2}, \quad \text{and} \quad p_1 = \frac{1}{C_1R_1} \tag{3.81}$$

Again, we observe that the pole and the zero may be set independently anywhere on the negative real axis, such as at the origin and at infinity. To illustrate that design is not unique, let us repeat the design of Example 3.5 for the circuit in Fig. 3.28.

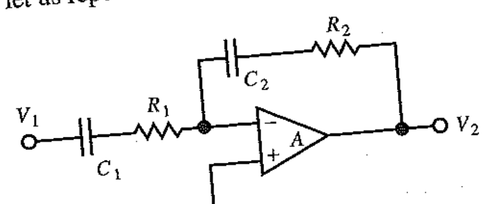


Figure 3.28 Alternative active realization of a bilinear function.

EXAMPLE 3.6

The transfer function of the circuit was given in Eqs. (3.72) and (3.73):

$$T(s) = -14.125\frac{s + 2\pi \times 830 \text{ Hz}}{s + 2\pi \times 13000 \text{ Hz}}. \tag{3.82}$$

Solution

Using no scaling, we obtain directly with Eq. (3.81)

$$\frac{R_2}{R_1} = 14.125, \quad \frac{1}{C_2R_2} = 2\pi \times 830 \text{ Hz}, \quad \text{and} \quad \frac{1}{C_1R_1} = 2\pi \times 13 \text{ kHz} \tag{3.83}$$

Choosing $C_1 = 10 \text{ nF} = 0.01\ \mu\text{F}$, results in the components

$$R_1 = \frac{1}{2\pi \times 13 \text{ kHz} \times 10 \text{ nF}} = 1.224 \text{ k}\Omega, \quad R_2 = 14.125R_1 = 17.289 \text{ k}\Omega$$

$$C_2 = \frac{1}{2\pi \times 830 \text{ Hz} \times 17.289 \text{ k}\Omega} = 11.09 \text{ nF}$$

The resulting circuit is seen in Fig. 3.25b. As the transfer function implemented by the two circuits in Fig. 3.25 is the same, we have demonstrated that design is not unique.

Further possibilities for the design of the circuit of Fig. 3.23 in selecting one capacitor and one resistor each for Z_1 and Z_2 are the following. We note that we could contemplate mixing the series and parallel connections of capacitor and resistor for the input branch and the feedback branch, i.e., place the parallel branch in the feedback path and the series connection in the input branch, or vice versa. This is shown in Fig. 3.29a and b. The transfer functions of these circuits are obtained most easily by noting from Eq. (3.67b) that for Fig. 3.29a:

$$T(s) = -\frac{1}{Z_1Y_2} \tag{3.84a}$$

Inserting the elements, we obtain

$$T(s) = -\frac{1}{Z_1Y_2} = -\frac{1}{[R_1 + 1/(sC_1)](G_2 + sC_2)} = -\frac{sC_1R_2}{(sC_1R_1 + 1)(sC_2R_2 + 1)} \tag{3.85a}$$

For Fig 3.29b we use

$$T(s) = -Y_1Z_2 \tag{3.84b}$$

so that

$$T(s) = -Y_1Z_2 = -(G_1 + sC_1)\left(R_2 + \frac{1}{sC_2}\right) = -\frac{(sC_1R_1 + 1)(sC_2R_2 + 1)}{sC_2R_1} \tag{3.85b}$$

We observe that both circuits in Fig. 3.29 lead to *second-order* functions. It suggests that the manner in which the capacitors are connected determines the order of the transfer function

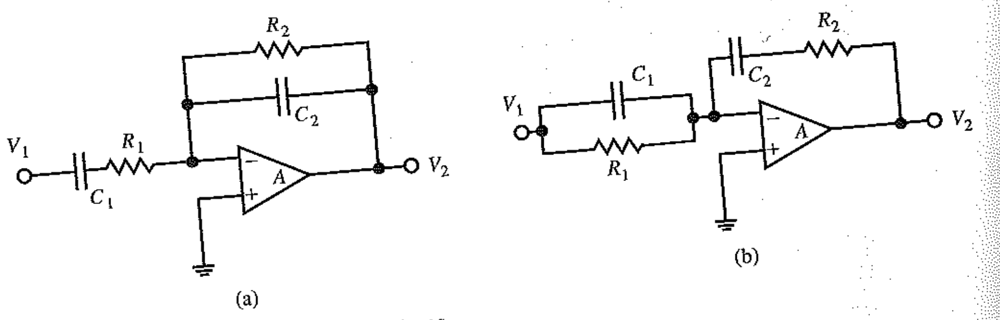


Figure 3.29 Two alternative filter structures.

realized. Equation (3.85a) has two negative real poles, and zeros at the origin and at infinity. The function is that of a bandpass as illustrated in Figs. 1.2c or 1.3c.

It is interesting to investigate the circuit in Fig. 3.29b more carefully. It is not useful because the (theoretical) poles at $s = 0$ and $s = \infty$ would imply infinite gain at the origin and at infinity. To understand the behavior suggested by Eq. (3.85b), we note that at $s = 0$ C_1 and C_2 are open circuits and the *opamp operates in open loop*. Further, V_1 is directly connected to the inverting input terminal of the opamp because at dc the current through R_1 is zero.[3] Thus the opamp would apply infinite gain to V_1, unconstrained by feedback. At very high frequencies, V_1 is again directly connected to the opamp (by C_1) and feedback is supplied via R_2. The gain $|R_2/Z_1|$ of the inverting amplifier becomes infinite because Z_1 appears to be zero (is a short circuit). In reality the gain will stay finite, of course, because, as was pointed out in Chapter 2, the electronic circuitry in the opamp will not support any voltage larger than the power supply. So, the increasing gain at $s \to 0$ and $s \to \infty$ will result in nonlinear opamp operation so that the result of Eq. (3.85b) is meaningless.

Let us point out in connection with this case that the student should always carefully examine a circuit when the analysis suggests a result that is physically meaningless. The cause for such unrealistic predictions is almost always found in poor or overly simplified device models being used for the mathematical analysis. In the case under discussion, we mistakenly assumed that, first, the opamp gain is infinite at high frequencies (neglecting the frequency dependence), and that, second, the opamp gain is linear (neglecting the limits placed on the magnitude of the output voltage). The result of Eq. (3.85b) was based on an unrealistic device model and is, therefore, not valid.

We have seen that for the appropriate choice of components the circuits in Figs. 3.24 and 3.28 can realize any bilinear function with negative real poles and zeros, including at the origin and at infinity, as long as the gain is inverting. Fortunately, in most applications this inversion, a phase shift of 180°, is unimportant. As we shall see later in our study, higher order filters are often designed by *cascading* low-order sections. We wish to make certain, therefore, that sections later in the cascade do not load the earlier sections. This means that their input impedances must be high, a condition not necessarily satisfied by the circuit in Fig. 3.23 whose input impedance was derived in Chapter 2 to be finite (it is equal to Z_1). On the other hand, the input impedance of a noninverting amplifier stage is essentially an open circuit. Let us, therefore, in the next section investigate the possibility of a circuit that delivers positive gain.

[3] Since both the opamp input and the capacitor C_2 are open circuits at dc, no dc current can flow through R_1.

3.4.2 Noninverting Opamp Circuits

Recalling our discussions from Chapter 2, we seek to utilize the noninverting amplifier of Fig. 2.11a with resistors replaced by impedances. This circuit is shown in Fig. 3.30. By direct analysis, or by analogy with Eq. (2.34), we obtain its transfer function as

$$\frac{V_2}{V_1} = \left(1 + \frac{Z_2}{Z_1}\right) \frac{1}{1 + \dfrac{1}{A(s)}\left(1 + \dfrac{Z_2}{Z_1}\right)} \tag{3.86}$$

Again we have shown the dependence on $A(s)$ because we must be mindful of its effect. For now let us assume the ideal case, $A(s) = \infty$, and consider the design possibilities of the function

$$T(s) = \frac{V_2}{V_1} = 1 + \frac{Z_2}{Z_1} = 1 + \frac{Y_1}{Y_2} = K\frac{e_n s + z_1}{e_d s + p_1} \tag{3.87}$$

In Eq. (3.87) we have temporarily introduced the two coefficients e_n and e_d in the numerator and the denominator. Both coefficients will be equal to either 1 or 0, depending on whether the s-terms are present or absent. This notation will help us state more clearly the conditions that the bilinear function must satisfy to be realizable by the circuit in Fig. 3.30.

The design will proceed by identifying possible impedances from

$$\frac{Z_2}{Z_1} = \frac{Y_1}{Y_2} = K\frac{e_n s + z_1}{e_d s + p_1} - 1 = \frac{(e_n K - e_d)\,s + Kz_1 - p_1}{e_d s + p_1} \tag{3.88}$$

As for the inverting-amplifier case, we recognize several possibilities: we assume normalized and scaled components and identify them by the subscript n. We may then set

$$Y_1 = sC_{1n} + G_{1n} = s\,(e_n K - e_d) + Kz_1 - p_1 \tag{3.89a}$$

and

$$Y_2 = sC_{2n} + G_{2n} = e_d s + p_1 \tag{3.89b}$$

When we work with impedances, we cannot simply equate the numerators and denominators of

$$\frac{Z_2}{Z_1} = \frac{(e_n K - e_d)\,s + Kz_1 - p}{e_d s + p_1}$$

because that would lead us to inductances. Rather, we use the step of dividing numerator and denominator by s to obtain

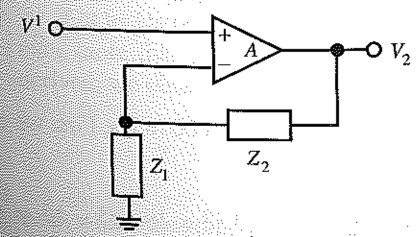


Figure 3.30 Noninverting amplifier with impedances in the feedback network.

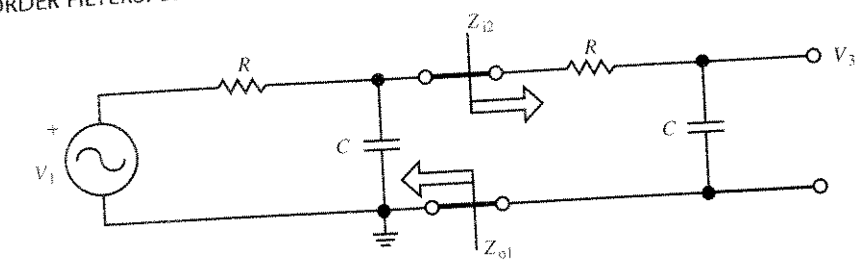


Figure 3.43 Cascade connection of two first-order passive lowpass sections.

$$Z_{i2} = R + \frac{1}{sC} \quad \text{and} \quad Z_{o1} = \frac{1}{sC + 1/R}$$

the transfer function

$$T(s) = \frac{V_3}{V_1} = \left(\frac{1}{sCR+1}\right)^2 \times \frac{R + \dfrac{1}{sC}}{\left(R + \dfrac{1}{sC}\right) + \left(\dfrac{R}{sCR+1}\right)}$$

$$= \frac{1}{(sCR+1)^2} \frac{(sCR+1)^2}{(sCR)^2 + 3sCR + 1} = \frac{1}{(sCR)^2 + 3sCR + 1} \tag{3.127}$$

This result can be confirmed by direct analysis of the circuit in Fig. 3.43. Comparing Eqs. (3.126) and (3.127), we see that the finite frequency-dependent input and output impedances changed the desired transfer function in a substantial way. Using the voltage follower of Fig. 2.26b, we can remedy the situation by *buffering* the sections as in Fig. 3.44 so that no current is drawn from the first section and the second section is driven by a nearly ideal voltage source.

To summarize the discussion, we repeat the important condition that must be satisfied if two circuits are to be connected in cascade: the trailing section must not load the leading section. The circuits to be cascaded can be of low order, high order, active, passive, or any desired combination; the loading restriction does not change. For designing high-order active filters, we shall see that the cascade connection normally consists of first- and second-order *active* building blocks because they can be cascaded directly with no need for buffering. Since active circuits usually have an opamp output as their output terminal [see e.g. Figs. (3.23), (3.30), and (3.34)], unbuffered cascading is possible because the opamp output resistance in a feedback network is very small (see Table 2.2). On the other hand, we can expect intuitively that *passive* circuits can generally not be connected in cascade without buffering because the condition (3.124) will rarely be satisfied. We mentioned at the end of Section 3.2 that the

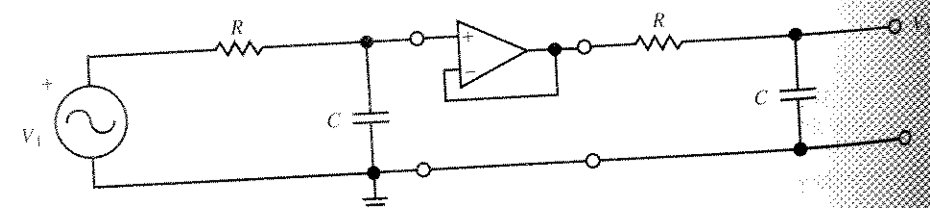


Figure 3.44 Two lowpass modules isolated by an opamp buffer.

possibility of loading a circuit without destroying the designed transfer function is a major benefit of active realizations. This advantage will come therefore to good use in the design of higher-order filters. For example, in the phase shifting network of Example 3.8, Fig. 3.38, the cascade connection causes no problems because the output impedance of the left amplifier is small compared to the input impedance of the following stage.

3.7 AND NOW DESIGN

We are now prepared to introduce some aspects of circuit design that will be amplified throughout the book. We do this in terms of some of the circuits analyzed earlier in this and in previous chapters, and we will make use of what we learned about opamp behavior. The design procedure may be summarized in the following steps:

Specification Some aspects of the required magnitude or phase response must be given along the frequency axis so that poles and zeros, i.e., the transfer function, can be identified. For example, if we describe filter requirements by a Bode plot, the break frequencies permit us to locate poles and zeros by a graphic trial-and-error procedure. We shall discuss in subsequent chapters a more rigorous mathematical process that provides the poles and zeros needed for the design without trial and error.

Requirement We require the complete circuit with element values in a practical range, consistent with the technology chosen.

Procedure The following steps are suggested:

1. Proceed from the specifications to determine the transfer function with its pole and zero locations. Poles and zeros are always needed before proceeding with filter design.
2. Select a circuit that promises to be able to satisfy the required magnitude or phase variation with frequency.
3. Use scaling and normalization, if desired, to reduce pole and zero locations to small numbers of the order on unity (eliminating high positive or negative powers of 10) and to be able to deal with dimensionless numbers for the components.
4. Determine the element values from the values of the poles and zeros.
5. If scaling was used, invert Step 3 to arrive at suitable practical component values.
6. Since a design problem never has a unique solution, investigate whether other circuit solutions exist that may be better suited to practical implementation in teems of component use, power consumption, or other factors.

The student may want to review some of the examples presented earlier in light of the six steps. A number of additional examples to point out various aspects of the design process follow.

EXAMPLE 3.11

Design a filter whose transfer characteristic satisfies the asymptotic Bode plot in Fig. 3.45a. It is that of a bandstop filter. There is no loss at high and low frequencies, but 20 dB attenuation is provided in the intermediate range $1000 \text{ rad/s} \leq \omega \leq 10000 \text{ rad/s}$.

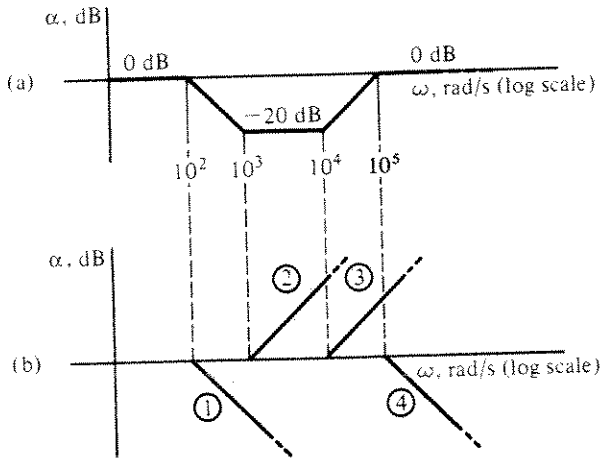


Figure 3.45 Specifications for Example 3.11.

Solution

The composite plot may be decomposed into four first-order factors as shown in Fig. 3.45b. Those marked 1 and 4 represent poles, whereas those marked 2 and 3 are zero factors. In other words, we see that

$$\alpha(\omega) = \alpha_0 - \alpha_1(\omega) + \alpha_2(\omega) + \alpha_3(\omega) - \alpha_4(\omega)$$

α_0 accounts for the possibility of a frequency-independent gain constant. From the given break frequencies and the 20-dB/decade slopes, we see that

$$T(j\omega) = K\frac{\left(1 + j\omega/10^3\right)\left(1 + j\omega/10^4\right)}{\left(1 + j\omega/10^2\right)\left(1 + j\omega/10^5\right)} \qquad (3.128)$$

Written in this form, it is clearly seen that when $\omega = 0$, $T(j0) = K$. From the figure we see that the low-frequency value of $\alpha(\omega) = \alpha_0 = 0$ dB, so that by Eq. (2.26), $K = 1$. Substituting s for $j\omega$ in Eq. (3.128) gives us the transfer function

$$T(s) = \frac{\left(s + 10^3\right)\left(s + 10^4\right)}{\left(s + 10^2\right)\left(s + 10^5\right)} \qquad (3.129)$$

With experience it will be possible to write $T(s)$ in this form directly from the asymptotic Bode plot, bypassing the intermediate steps.

We next write $T(s)$ as a product of bilinear functions. The choice is arbitrary, but one possibility is

$$T(s) = T_1(s)T_2(s) = \frac{s + 10^3}{s + 10^2} \times \frac{s + 10^4}{s + 10^5} \qquad (3.130)$$

For a circuit realization of T_1 and T_2, we next decide to use the inverting opamp circuit of Fig. 3.24. Using the formulas for element values give in Eq. (3.71), we obtain for the first section

$$\frac{1}{C_{11}R_{11}} = 1000 \text{ rad/s} \quad \text{and} \quad \frac{1}{C_{21}R_{21}} = 100 \text{ rad/s}$$

and for the second section

$$\frac{1}{C_{12}R_{12}} = 10,000 \text{ rad/s} \quad \text{and} \quad \frac{1}{C_{22}R_{22}} = 100,000 \text{ rad/s}$$

It is convenient to choose all capacitors of the same value; let us select $C = 0.01\ \mu$F. Then we obtain

$$R_{11} = \frac{1}{1000 \text{ rad/s} \times 0.01\ \mu\text{F}} = 100\ \text{k}\Omega \quad \text{and} \quad R_{21} = \frac{1}{100 \text{ rad/s} \times 0.01\ \mu\text{F}} = 1\ \text{M}\Omega$$

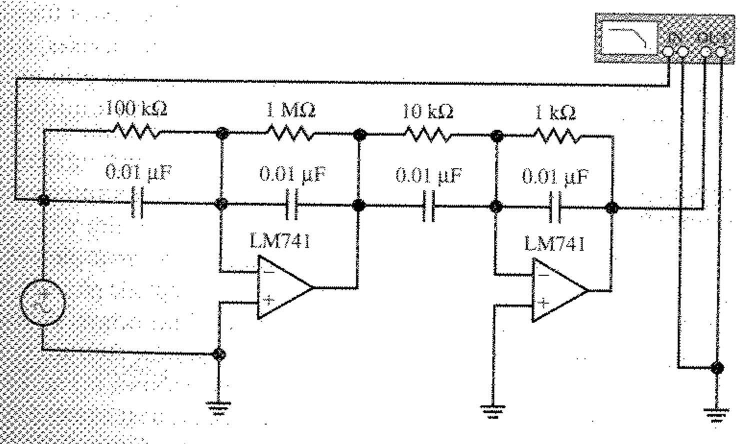
$$R_{12} = \frac{1}{10{,}000 \text{ rad/s} \times 0.01\ \mu\text{F}} = 10\ \text{k}\Omega \quad \text{and} \quad R_{22} = \frac{1}{100{,}000 \text{ rad/s} \times 0.01\ \mu\text{F}} = 1\ \text{k}\Omega$$

The element values that result are shown in Fig. 3.46a and the design is complete. A characteristic of design that again becomes apparent is that there is no unique solution. If, e.g., the spread of resistor sizes in Fig. 3.46a is too large, then we begin the design process again and obtain a different result. This process may be repeated several times until a satisfactory solution is obtained.

To consider the effect of a real opamp on the transfer function of Eq. (3.130) we insert the specified poles and zeros of the two sections into Eq. (3.111):

$$T_1(s) = -\frac{s + 10^3}{2s^2/\omega_t + s\,(1 + 1100/\omega_t) + 100} \quad \text{and}$$

$$T_2(s) = -\frac{s + 10^4}{2s^2/\omega_t + s\left(1 + 11 \times 10^4/\omega_t\right) + 10^5} \qquad (3.131)$$



(a)

Figure 3.46 (a) Circuit realizing Eq. (3.128) of Example 3.11 and test set-up; (b) measured circuit performance with LM741 opamps (cursor readout is 100 kHz, 1.495 dB); (c) measured circuit performance with HA2542-2 opamps (cursor at 540.3 Hz, −19.17 dB). (The frequency scale of the Bode Plotter is set from 1 Hz to 1 MHz; the vertical axis goes from −20 dB to +10 dB.)

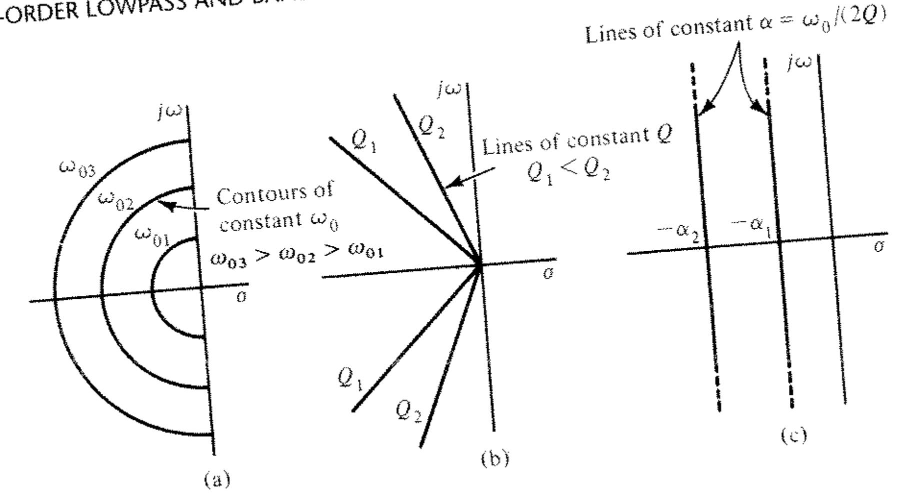


Figure 4.3 Contours in the s-plane.

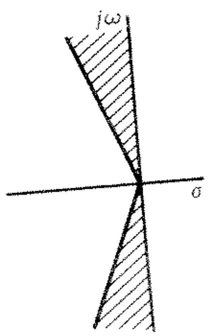
cles of radius ω_0 with their centers at the origin, as shown in Fig. 4.3a. From Eq. (4.13) we see that lines of constant Q are lines of constant angle ψ, as shown in Fig. 4.3b. Finally, lines of constant ratio $\omega_0/(2Q)$ are lines parallel to the imaginary axis, as shown in Fig. 4.3c.

In circuit design we will ordinarily deal with Q values greater than 1. This has implications with respect to pole positions. From Eq. (4.13) we make the following tabulation:

Q	ψ (degrees)
0.707	45
1	60
2	75.52
5	84.3
20	88.5
100	89.7

Hence we conclude that we will normally be interested in a small sector of the s-plane, which is shaded in Fig. 4.4. Observe that when Q is greater than 5, then Eq. (4.12) for β simplifies to $\beta \approx \omega_0$ with an error less than 1%.

Figure 4.4 Sectors of usual pole locations.



EXAMPLE 4.1

The two poles of a given $T(s)$ are located in the s-plane on lines of slope ± 2, as shown in Fig. 4.5. (a) Determine an expression for the Q of these poles. (b) Express the pole locations $-\alpha \pm j\beta$ in terms of ω_0.

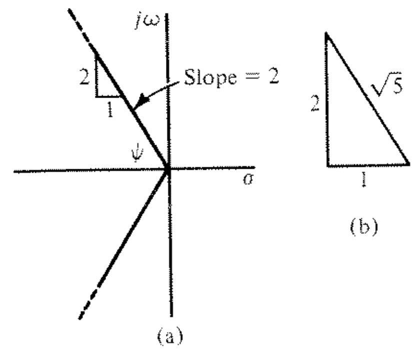


Figure 4.5 (a) Identifying Q in Example 4.1 from the angle (b) of their locus in the s plane.

Solution

Since the slope of the s-plane line is 2, then $\tan\psi = 2$. For this angle, shown in Fig. 4.5b, $\cos\psi = 1/\sqrt{5}$. Combining this result with Eq. (4.13), we see that

$$\cos\psi = \frac{1}{2Q} = \frac{1}{\sqrt{5}} \tag{4.14}$$

so that

$$Q = \frac{\sqrt{5}}{2} \tag{4.15}$$

Once Q is determined, then Eqs. (4.9) and (4.12) give the pole location as

$$\alpha = \frac{1}{2Q}\omega_0 = \frac{\omega_0}{\sqrt{5}}, \qquad \beta = \omega_0\sqrt{1 - \frac{1}{4Q^2}} = \frac{2}{\sqrt{5}}\omega_0 \tag{4.16}$$

If ω_0 is also specified, then the pole locations are fixed.

Returning to the circuit of Fig. 4.1b and the associated transfer function which described it, Eq. (4.2), we see that the three element values R, L, and C completely specify $T(s)$. But we have now shown that the two parameters Q and ω_0 also specify $T(s)$, as given by Eq. (4.7). We now have to relate Q and ω_0 to the magnitude and phase responses, which we will in turn relate to filter specifications.

4.2 THE SECOND-ORDER CIRCUIT

The transfer function for the lowpass filter derived as Eq. (4.7) was written in a normalized form such that $T(j0) = 1$. A more general form for $T(s)$ in active circuits will recognize the possibility of gain and also that the associated circuit may be inverting or noninverting. Such a transfer function is

$$T(s) = \frac{\pm H\omega_0^2}{s^2 + (\omega_0/Q)s + \omega_0^2} \tag{4.17a}$$

and can be represented by the block diagram in Fig. 4.6. Next we do what will be done frequently in the chapters to follow: we scale the frequency by dividing s by ω_0, i.e., we use the normalized frequency $s_n = s/\omega_0$. To accomplish this step, we divide numerator and denominator of Eq. (4.17a) by ω_0^2,

$$T(s) = \frac{\pm H}{(s/\omega_0)^2 + (1/Q)(s/\omega_0) + 1} = \frac{\pm H}{s_n^2 + (1/Q)s_n + 1} \tag{4.17b}$$

Note that the result obtained is the same as if we had simply set $\omega_0 = 1$ in Eq. (4.17a). When in the discussions to follow we refer to "setting $\omega_0 = 1$" we mean the normalization (or scaling) just discussed. However, to keep the notation simple, we shall in the following drop the subscript n from the frequency parameter s and remember that in most filter work we deal in normalized frequencies. The context will always make clear what is meant. Thus, in what follows, s is a normalized frequency. We also choose the negative sign in Eq. (4.17), meaning that we anticipate an inverting realization of the transfer function. Then Eq. (4.17b) becomes

$$T(s) = \frac{V_L}{V_1} = \frac{-H}{s^2 + (1/Q)s + 1} \tag{4.18}$$

We wish to manipulate this equation until it has a form that can be identified with the simple circuits studied previously. We begin by rewriting Eq. (4.18) as

$$\left(s^2 + \frac{1}{Q}s + 1\right)V_L = -HV_1 \tag{4.19}$$

Let us take a small excursion into the time domain to see what steps to take next and help us develop a circuit whose output is V_L. We remind ourselves that s in the Laplace domain represents the differentiation operator, so that Eq. (4.19) is the Laplace transform of a second-order differential equation,

$$\frac{d^2 v_L(t)}{dt^2} + \frac{1}{Q}\frac{dv_L(t)}{dt} + v_L(t) = -Hv_1(t)$$

Thus, to determine the output voltage $v_L(t)$ from the input $v_1(t)$ we need to perform two integrations. But since dividing by s is an easier operation than integration ($1/s$ is the integration operator), let us return to the frequency domain and recast Eq. (4.19) in a form that lets us identify integration, $1/s$. We obtain

$$s(s + 1/Q)V_L = -(HV_1 + V_L)$$

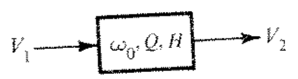


Figure 4.6 Block diagram of second-order section.

or

$$sV_L = -\frac{1}{s + 1/Q}(HV_1 + V_L) = V_B \tag{4.20}$$

We see that V_L is obtained by integrating the voltage on the right-hand side, identified as V_B:

$$V_L = \frac{1}{s}V_B = \frac{1}{s}\left[-\frac{1}{s + 1/Q}(HV_1 + V_L)\right] \tag{4.21}$$

To further resolve the operation in the brackets, we rewrite Eq. (4.20) as

$$(s + 1/Q)V_B = -HV_1 - V_L$$

or

$$V_B = -\frac{1}{s}\left(HV_1 + V_L + \frac{1}{Q}V_B\right) = -\frac{1}{s}V_H \tag{4.22}$$

Equation (4.22) says that V_B is obtained by integrating, with a sign inversion, the voltage V_H, which in turn is obtained by summing three scaled voltages,

$$V_H = HV_1 + V_L + \frac{1}{Q}V_B \tag{4.23}$$

Thus we have identified the double integration to obtain V_L

$$V_L = \left(\frac{1}{s}\right) \times \left[\left(-\frac{1}{s}\right) \times V_H\right] \tag{4.24}$$

with V_H obtained from the sum in Eq. (4.23). A block diagram depicting our result is shown in Fig. 4.7a. We see that the summing node realizes the weighted sum of the three voltages V_1, V_L, and V_B as prescribed in Eq. (4.23), followed by an inverting integrator to produce V_B as in Eq. (4.22) and a final noninverting integrator to obtain the lowpass output V_L as Eq. (4.21) demands. It is interesting to observe the functions realized at the other outputs, V_B and V_H. V_B is seen to be obtained by multiplying V_L by s and V_H by multiplying V_B by $-s$. With Eq. (4.18), the results are

$$V_L = \frac{-H}{s^2 + (1/Q)s + 1}V_1 = T_L V_1 \tag{4.25a}$$

$$V_B = \frac{-Hs}{s^2 + (1/Q)s + 1}V_1 = T_B V_1 \tag{4.25b}$$

$$V_H = \frac{Hs^2}{s^2 + (1/Q)s + 1}V_1 = T_H V_1 \tag{4.25c}$$

that is, at the three nodes the block diagram realizes not only a lowpass function T_L, but also what will be recognized as a *bandpass* T_B and a *highpass* T_H. Note that V_B equals HQ at $s = \pm j(\omega = \omega_0 = 1)$ and goes to zero at low and at high frequencies as was sketched in Fig. 1.2c. The voltage V_H at the highpass output equals zero at $s = 0$ and then increases with frequency, reaching HQ at $s = \pm j$ and H at $s = \infty$.

Having developed a simple block diagram to realize the transfer functions of Eq. (4.25), we need to consider next how to implement the circuit with real components. To this end we return to the circuits we developed in Chapters 2 and 3 and recall that we were able to realize an ideal inverting integrator, Fig. 3.2a, but not an ideal noninverting integrator. But two inverting integrators cannot be used directly because we must ensure stability, which requires negative feedback in all loops. To solve this difficulty, we introduce a minor modification into the block diagram of Fig. 4.7a and realize a noninverting integrator as a cascade connection of an inverting integrator and an inverter, $(1/s) = (-1/s) \times (-1)$, as is shown in Fig. 4.7b. This approach provides the additional advantage of letting us realize a *non*inverting lowpass function at the output $-V_L$. We can now use two of the inverting integrators of Fig. 3.2a, T_2 in Fig. 4.8, followed by the inverter of Fig. 2.17a with $R_1 = R_2 = 1$, T_3 in Fig. 4.8. At the input we need the summer of Fig. 2.29a, which we can combine (merge) with the integrator by replacing R_F by a resistor and capacitor in parallel; this circuit is shown as T_1 in Fig 4.8. The module T_1 realizes, assuming an ideal opamp, $V_B/A \approx 0$,

$$1 \times V_L + H \times V_1 + (1/Q + s) \times V_B \approx 0$$

that is, as required in Eq. (4.20),

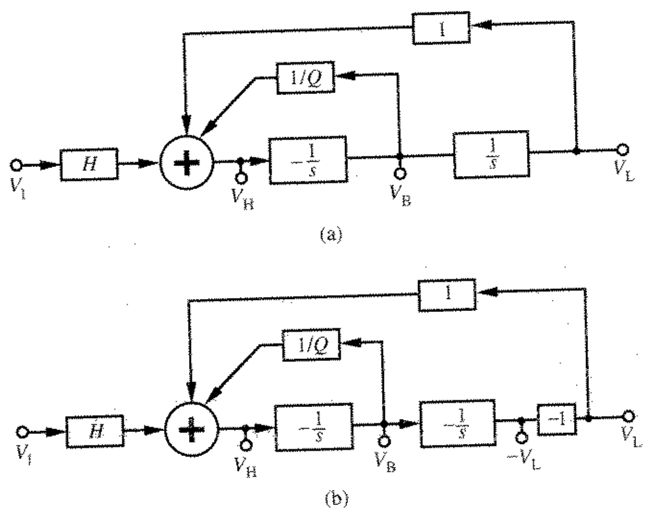


Figure 4.7 Block diagram (two-integrator loop) to implement Eq. (4.18); (a) using an inverting and a noninverting integrator; (b) using two inverting integrators and an inverter.

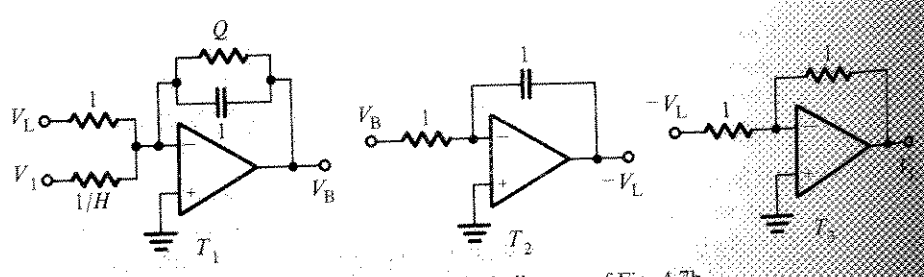


Figure 4.8 Circuit modules to implement the block diagram of Fig. 4.7b.

$$V_B = -\frac{1}{s + 1/Q}(V_L + HV_1) \tag{4.26}$$

Notice that the feedback resistor across the ideal (lossless) integrator makes the integrator lossy. Combining this equation with

$$-V_L = -\frac{1}{s}V_B \quad \text{and} \quad V_L = (-1)(-V_L)$$

and making the connections suggested by the labels at the terminals of the three modules results in the full circuit in Fig. 4.9. This filter is the so-called *Tow–Thomas* biquad (Tow, 1968; Thomas, 1971). To see how the elements of the circuit enter the transfer function explicitly, let us label them as in Fig. 4.10. Routine analysis results in the lowpass and bandpass functions

$$T_L(s) = -\frac{1/(R_3R_4C_1C_2)}{s^2 + s/(R_1C_1) + 1/(R_2R_4C_1C_2)} \tag{4.27a}$$

$$T_B(s) = (-sC_2R_4) \times [-T_L(s)] = -\frac{(R_1/R_3) \cdot s/(R_1C_1)}{s^2 + s/(R_1C_1) + 1/(R_2R_4C_1C_2)} \tag{4.27b}$$

The circuit cannot realize the highpass output because we have chosen to merge the summing block with the first integrator. If the highpass output is required, an additional opamp is

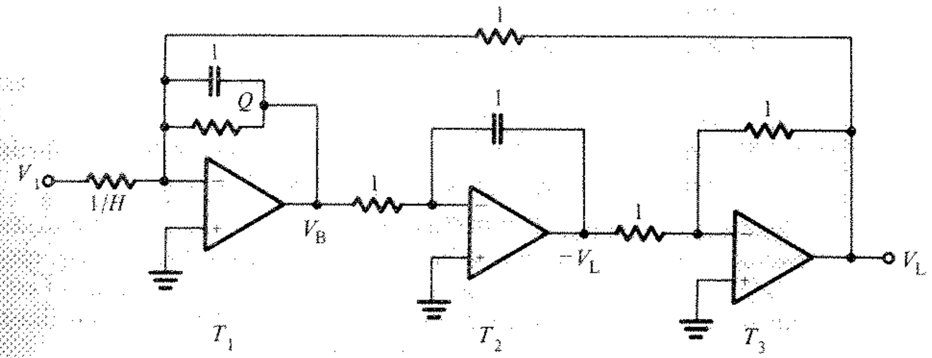


Figure 4.9 The Tow–Thomas biquad with normalized elements.

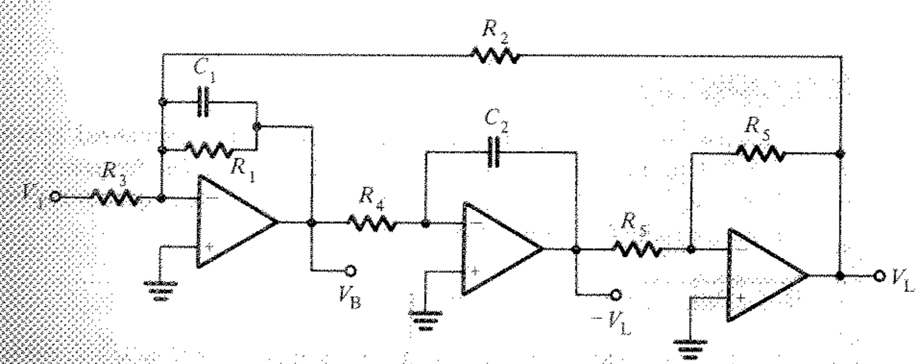


Figure 4.10 The Tow–Thomas biquad.

needed to permit realizing the summing operation (Fig. 2.29a) and the integration (Fig. 3.26a) separately.

Comparing Eq. (4.27a) to the standard form of Eq. (4.17), we identify the appropriate coefficients with the element values as

$$\omega_0^2 = \frac{1}{R_2 R_4 C_1 C_2}, \quad Q = \frac{R_1}{\sqrt{R_2 R_4}}\sqrt{\frac{C_1}{C_2}}, \quad \text{and} \quad H = \frac{R_2}{R_3} \tag{4.28}$$

We can now determine the element values to satisfy the given design parameters. This is a typical situation in active filter design: we have more components (here six) than parameters (here three). We will therefore select arbitrarily three of the components and then examine the consequences on the remaining three. Since we used frequency scaling ($\omega_0 = 1$) and intend to use magnitude scaling as well, we make the following choices:

$$C_1 = C_2 = 1 \quad \text{and} \quad R_4 = 1$$

and obtain from Eq. (4.28)

$$R_1 = Q, \quad R_2 = 1, \quad \text{and} \quad R_3 = 1/H$$

These element values give us exactly the circuit previously derived in Fig. 4.9.

An important property of the biquad circuit is that it can be *orthogonally* tuned. By this we mean that

1. R_2 can be adjusted to a specified value of ω_0.
2. R_1 can then be adjusted to give the specified value of Q without changing ω_0, which has already been adjusted.
3. Finally R_3 can be adjusted to give the desired value of H or gain for the circuit, without affecting either ω_0 or Q, which have already been set.

These steps are often called the *tuning algorithm*. This algorithm provides for orthogonal tuning. If this tuning is not possible, then the tuning is called *iterative*, meaning that we try to adjust successively each of the tuning elements until all specifications are met. Orthogonal tuning is always much preferred, especially when the filter is to be produced on a production line with a laser used to adjust each circuit element value.

An example will help us understand the design process:

EXAMPLE 4.2

A lowpass filter is to be designed whose poles in the normalized s-plane are located at $-0.577 \pm j0.8165$. The dc ($\omega \to 0$) gain is to be 2. The frequency is scaled by $\omega_0 =$ 20,000 rad/s ($f_0 = 3183$ Hz). Find the values of the pole frequency and pole quality factor and design a circuit to realize the specifications.

Solution

From Eqs. (4.10) and (4.11) we find $\omega_0 = 1$ and $Q = 0.877 = \sqrt{3}/2$. Also, from Eq. (4.17) $H = 2$. Let us choose $C_1 = C_2 = C = 0.01\ \mu$F and the normalizing resistor $R = 10$ kΩ. Choosing $R_2 = R_4$ in Eq. (4.28) with the given value of ω_0 we find

$$R_2 = R_4 = \frac{1}{\omega_0 C} = \frac{1}{2 \times 10^4 \text{s}^{-1} \times 10^{-8}\text{F}} = 5\ \text{k}\Omega$$

and

$$R_3 = \frac{R_2}{H} = 2.5\ \text{k}\Omega, \quad R_1 = \sqrt{R_2 R_4}\,Q = 5 \times 10^3\ \Omega \times 0.877 = 4.33\text{k}\Omega$$

The two resistors of the inverter are also chosen as 10 kΩ. The circuit and its test performance are shown in Fig. 4.11. We notice that the circuit is a lowpass filter with a dc gain of $H = 2$, i.e., 6 dB as specified. The passband corner is at ≈ 3.7 kHz (a little higher than 20,000 rad/s divided by 2π, see Fig. 4.13); it has a small peak near the passband edge, and the attenuation increase for frequencies above 3.7 kHz is 20 dB per decade (6 dB per octave), because $T(s)$ decreases as $1/s^2$. We also observe an unexpected dip in the attenuation curve at approximately 350 kHz that can be shown to be caused by higher-order poles in the opamp model.

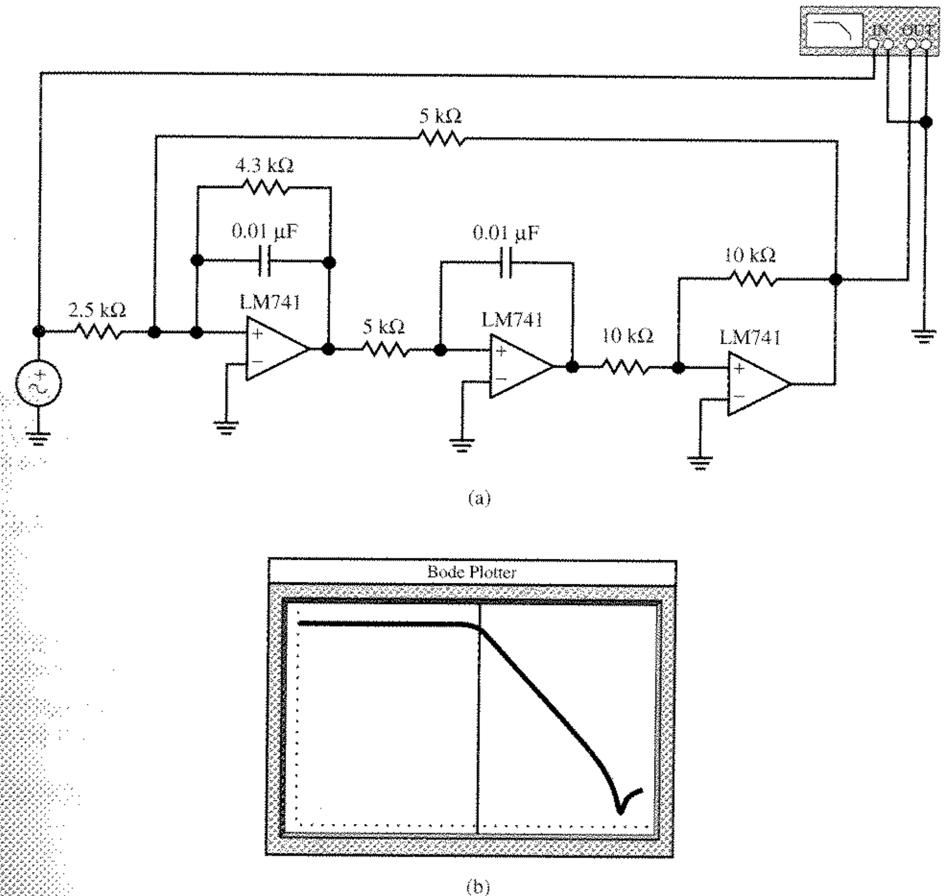
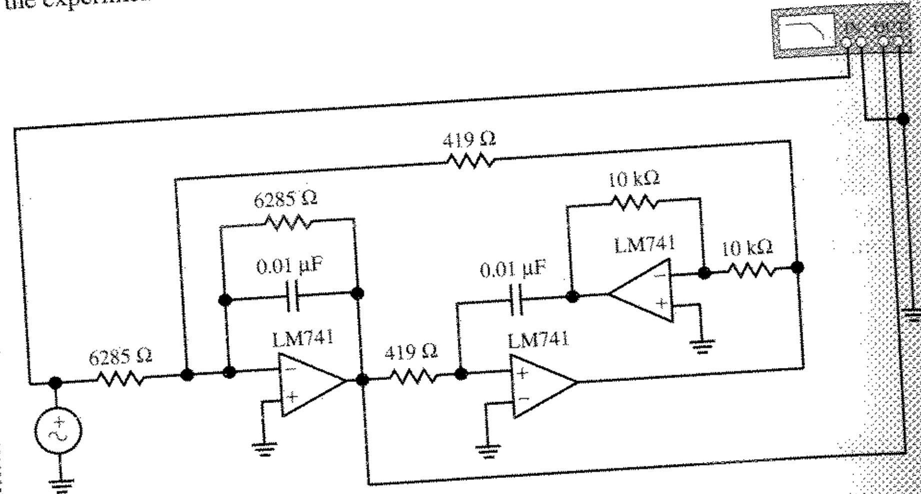
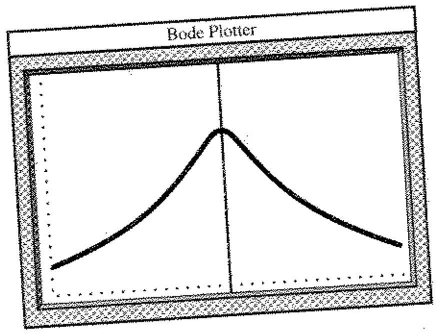


Figure 4.11 (a) Circuit for Example 4.2 and (b) its performance. (Bode Plotter scales: 10 Hz to 700 kHz; −100 dB to +10 dB; cursor at 3.70 kHz, 3.10 dB.)

given in Fig. 4.30b. Although we based the design on ideal amplifiers, the performance with real opamps is nearly as predicted: The experimental midband gain is 1.24 rather than 1, and Q equals 14.3 rather than 15 as specified. As in the previous two-integrator loop structures, the experimental center frequency equals 36.9 kHz.



(a)



(b)

Figure 4.30 (a) Filter designed for Example 4.6; (b) experimental performance. (Bode Plotter scales: 30 to 45 kHz; −20 dB to +10 dB; cursor at 36.9 kHz, 1.86 dB.)

This example demonstrates that the Åckerberg–Mossberg biquad is a filter that can be designed reliably for higher frequencies and large quality factors without incurring large unpredictable Q and gain errors. Frequency deviations are, of course, not affected by the circuit structure because, by Eq. (4.89), q cancellations do not influence f_0.

We have now analyzed in great detail the two-integrator-loop (Tow–Thomas) biquad in Fig. 4.10 and a number of modifications. With the help of our study of integrators, we arrived at the alternate implementations. The understanding gained showed us that finite integrator losses are the cause of errors in biquad performance. In particular, we discovered that at least the errors in quality factor could be eliminated entirely by pairing an

integrator with a positive loss term with one with a negative loss term. This resulted in the Åckerberg–Mossberg biquad of Fig. 4.29. Observe that the improved performance over that of the Tow-Thomas biquad in Fig. 4.10 is obtained at no additional cost. It requires only a wiring change. We will, therefore, in the remainder of this book deal only with the Åckerberg–Mossberg biquad whenever we need to use a two-integrator loop circuit. Nevertheless, the literature on active filters contains a great number of second-order circuits developed for a variety of applications. In the next section we shall study a few of them to present the student with a choice of circuits that may be called on to realize a design requirement.

4.5 OTHER BIQUADS

In addition to the three-amplifier biquads of Section 4.4, engineers have invented biquads that use two opamps or even only one opamp. The number of opamps does, of course, affect cost and power consumption, but the main differences between different circuits are their sensitivity to component tolerances, to be discussed in Chapter 12, and their versatility in being able to realize different transfer functions.

4.5.1 Sallen–Key Circuits

Sallen–Key filters were among the first active filters that appeared in the literature (Sallen and Key, 1955). This reference contains a whole catalog of possible structures that permit the realization of various types of transfer functions. One such structure to realize a lowpass function is shown in Fig. 4.31a. To analyze the circuit for its transfer function we obtain first by voltage division

$$\frac{V_2}{V_-} = 1 + \frac{R_B}{R_A} = K \tag{4.96}$$

and write node equations at nodes V_a and V_b:

$$(sC_2 + G_2)\,V_a = G_2 V_b \tag{4.97}$$

$$V_b\,(G_1 + G_2 + sC_1) = G_1 V_1 + sC_1 V_2 + G_2 V_a \tag{4.98}$$

For an ideal opamp we have $V_a - V_- = 0$ and we can solve the previous three equations for the transfer function V_2/V_1. Let us instead assume a real opamp with finite gain $A(s)$ and use $V_a - V_- = V_2/A$ to be able to investigate the effects of A. We next solve Eq. (4.98) for V_b and substitute the result into Eq. (4.97):

$$\left(sC_2 + G_2 - \frac{G_2^2}{G_1 + G_2 + sC_1}\right) V_a = G_2 \frac{G_1 V_1 + sC_1 V_2}{G_1 + G_2 + sC_1}$$

Solving for V_a, we obtain

$$V_a = \frac{G_1 G_2 V_1 + sC_1 G_2 V_2}{s^2 C_1 C_2 + s[C_2 (G_1 + G_2) + C_1 G_2] + G_1 G_2}$$

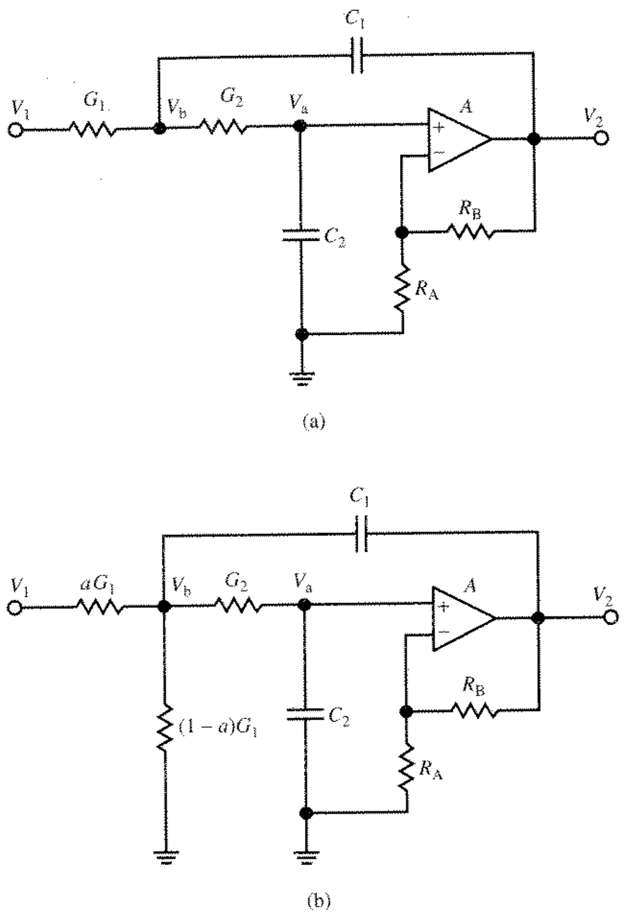


Figure 4.31 Sallen–Key lowpass filter; (a) dc gain $H = K$; (b) dc gain $H = aK$.

which with Eq. (4.96) and $V_a - V_- = V_2/A$ results in

$$T(s) = \frac{V_2}{V_1} = \frac{KG_1G_2\dfrac{1}{1+K/A}}{s^2C_1C_2 + s\left[C_2(G_1+G_2) + C_1G_2\left(1 - K\dfrac{1}{1+K/A}\right)\right] + G_1G_2}$$

To derive the design equations, let us assume for now an ideal opamp, $A = \infty$. We then have the lowpass function

$$T(s) = \frac{V_2}{V_1} = \frac{KG_1G_2}{s^2C_1C_2 + s[C_2(G_1+G_2) + C_1G_2(1-K)] + G_1G_2}$$

For convenience we set $C_1 = C_2 = C$ to obtain

$$T(s) = \frac{KG_1G_2/C^2}{s^2 + s[G_1 + G_2(2-K)]/C + G_1G_2/C^2} = \frac{H\omega_0^2}{s^2 + s\omega_0/Q + \omega_0^2} \quad (4.101)$$

We have again expressed the function in its standard form as in Eq. (4.17a) that lets us identify how the three filter parameters are expressed in terms of components:

$$\omega_0^2 = \frac{G_1G_2}{C^2} \quad (4.102)$$

$$Q = \frac{\sqrt{G_1G_2}}{G_1 + G_2(2-K)} \quad (4.103)$$

$$H = K > 1 \quad (4.104)$$

In selecting the four elements C, G_1, G_2, and K to realize the three filter parameters we have some freedom of choice. Since, in practice, the available capacitor values are usually limited, let us choose a convenient value for C. Also, let us assume that the dc gain is not important and can be fixed at $H = K$. Then we may select $R_1 = R_2 = R$ and obtain with

$$Q = \frac{1}{3-K} \quad (4.105)$$

the element values

$$R = \frac{1}{\omega_0 C} \quad \text{and} \quad K = 3 - \frac{1}{Q} = 1 + \frac{R_B}{R_A}, \quad \text{i.e.,} \quad R_B = (2 - 1/Q)R_A \quad (4.106)$$

R_A is arbitrary and can be chosen equal to R to minimize the number of different resistor values.

EXAMPLE 4.7

Design a Sallen–Key lowpass filter with $f_0 = 12.5$ kHz and no peaking. The dc gain is not specified. Use an LM741 opamp.

Solution

According to Fig. 4.13 a value $Q = 0.707$ is required to avoid peaking. Choosing $C = 0.01\mu$F results in

$$R_1 = R_2 = R_A = \frac{1}{2\pi \times 12.5 \times 10^{+4} \times 10^{-8}}\ \text{k}\Omega = 1.273\ \text{k}\Omega,$$
$$R_B = R_A(2 - 1/0.707) = 746\ \Omega$$

from Eq. (4.106). Figure 4.32 shows the circuit and its performance. The dc gain equals 4 dB as designed, and $f_0 = 12.5$ kHz. The dip and subsequent rise in gain at high frequencies are caused by higher-order opamp dynamics.

values of sensitivity are desirable, whereas circuits with large sensitivities ought to be avoided. Analogous expressions are true for the parameters of the transfer function:

$$S_R^Q = \frac{\partial Q/Q}{\partial R/R}, \quad S_R^{\omega_0} = \frac{\partial \omega_0/\omega_0}{\partial R/R} \tag{12.4b}$$

These equations are general; let us for the purpose of a general discussion of sensitivity label the function or parameter in question Y and the component x (see Fig. 12.4). Then the sensitivity expression becomes

$$S_x^Y = \frac{\partial Y/Y}{\partial x/x} = \frac{x}{Y}\frac{\partial Y}{\partial x} \tag{12.6}$$

Recall from differential calculus that

$$d(\ln u) = \frac{du}{u} \tag{12.7}$$

Then we may rewrite the sensitivity also as

$$S_x^Y = \frac{x}{Y}\frac{\partial Y}{\partial x} = \frac{\partial \ln Y}{\partial \ln x} \tag{12.8}$$

Two important observations may be made at this point. First we note that since sensitivity is computed from the slope of the function Y, we need to be consistent on our understanding of where the slope is measured: the derivative is evaluated at the *nominal* component value (at x_0 in Fig. 12.4). Second, we note from Eq. (12.4a) that in general sensitivity is a function of frequency if Y depends on frequency. For example, the sensitivity of a transfer function magnitude to a component R at the nominal value R_0 equals

$$S_R^{|T(j\omega, R_0)|} = \frac{R}{|T(j\omega, R)|}\frac{\partial |T(j\omega, R)|}{\partial R}\bigg|_{R=R_0} \tag{12.9}$$

Clearly, it depends on frequency. So we must make certain that we agree at which frequency it is measured and evaluate the sensitivity at or close to the operating frequency. It is as a rule of little relevance that one's design has very low sensitivities at dc when the circuit is to operate in the range of a few hundred kHz.

The problem of interest to Bode when he introduced the concept of sensitivity (Bode 1945) was the change in a transfer function T when one of the elements in the transmission system, an amplifier, was likely to suffer large changes. If the system is represented as in Fig. 12.5a where T_2 represents the amplifier, the overall transfer function is

$$T = T_1 T_2$$

It is clear that T_2 directly affects the transmission; its sensitivity to T_2 is

$$S_{T_2}^T = \frac{T_2}{T}\frac{\partial T}{\partial T_2} = \frac{T_2}{T}T_1 = \frac{T_1 T_2}{T_1 T_2} = 1$$

From this we see that a 1% change in T_2 results in a 1% change in T as anticipated. If feedback is introduced as in Fig. 12.5b then, as studied in Chapter 2, Eq. (2.77),

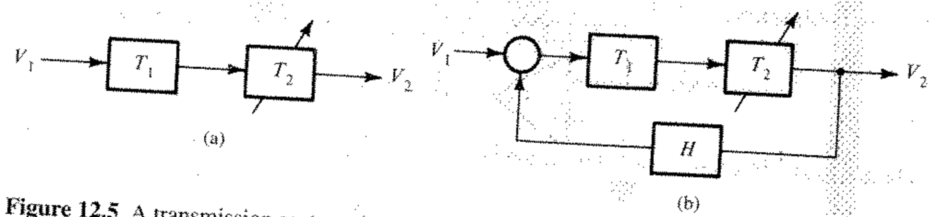


Figure 12.5 A transmission system: (a) without feedback; (b) with feedback.

$$T = \frac{T_1 T_2}{1 + H T_1 T_2} \tag{12.12}$$

and the sensitivity function is found by applying Eq. (12.8),

$$S_{T_2}^T = \frac{T_2}{T}\frac{\partial T}{\partial T_2} = \frac{T_2}{T}\frac{T_1(1 + HT_1T_2) - T_1T_2HT_1}{(1 + HT_1T_2)^2} = \frac{1}{1 + HT_1T_2} \tag{12.13}$$

Now if we select the parameters such that the *loop gain* $HT_1T_2 \gg 1$, the transfer function is

$$T \approx \frac{1}{H}$$

and the sensitivity becomes

$$S_{T_2}^T \approx \frac{1}{HT_1T_2} \ll 1$$

For example, if the loop gain is 1000, the sensitivity is 10^{-3} compared to the value 1 for the open-loop system. The result of this analysis led to the widespread use of feedback in transmission systems.

EXAMPLE 12.1

Apply sensitivity analysis to the inverting amplifier of Chapter 2, shown for convenience in Fig. 12.6.

Solution

We have from Eq. (2.55),

$$T = \frac{V_2}{V_1} = -\frac{R_2/R_1}{1 + (1 + R_2/R_1)/A} = -\frac{G_0}{1 + (1 + G_0)/A}$$

where we have labeled the amplifier's low-frequency gain G_0. The sensitivity of T with respect to R_1 is

$$S_{R_1}^T = \frac{R_1}{T}\frac{\partial T}{\partial R_1} = \frac{R_1}{T}\frac{\partial}{\partial R_1}\left[\frac{-R_2}{R_1 + (R_1 + R_2)/A}\right] = -\frac{R_1}{T}\frac{-R_2(1 + 1/A)}{[R_1 + (R_1 + R_2)/A]^2}$$

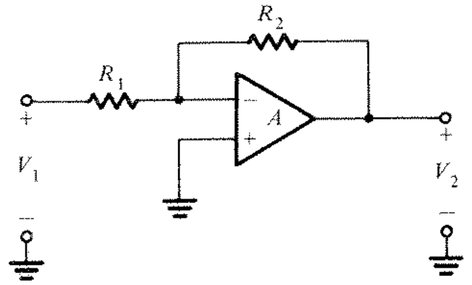


Figure 12.6 Inverting amplifier based on an opamp.

which we evaluate at $A \to \infty$:

$$S_{R_1}^T = \left.\frac{R_1(1+1/A)}{R_1+(R_1+R_2)/A}\right|_{A\to\infty} = -1$$

Similarly, we obtain

$$S_{R_2}^T = \frac{R_2}{T}\frac{\partial T}{\partial R_2} = \frac{R_2}{T}\frac{R_1+(R_1+R_2)/A - R_2/A}{[R_1+(R_1+R_2)/A]^2} = \left.\frac{R_1+R_1/A}{[R_1+(R_1+R_2)/A]}\right|_{A\to\infty} = 1$$

These results show that a 1% change in the resistors R_1 and R_2 causes, respectively, a −1% or +1% change in G_0. Since accurate and stable resistors are available, this value of sensitivity is acceptable. To evaluate the sensitivity to the opamp gain A, we compute

$$S_A^T = \frac{A}{T}\frac{\partial}{\partial A}\left[\frac{-G_0}{1+(1+G_0)/A}\right] = \frac{-(1+G_0)}{(A+1+G_0)}$$

Since usually $A \gg 1+G_0$ in the frequency range of interest, the sensitivity is approximately

$$S_A^T \approx \frac{1}{A}(1+G_0)$$

This is a very small number for large A. It demonstrates that the amplifier gain is approximately equal to $G_0 = -R_2/R_1$, independent of A, as long as the opamp gain A is large. However, if R_2 is removed ($R_2 = \infty$) so that there is no feedback, the sensitivity becomes

$$S_A^T = \left.\frac{R_1+R_2}{(A+1)R_1+R_2}\right|_{R_2\to\infty} \approx 1$$

and changes in A translate directly into changes of G_0. Whereas sensitivities of the order of unity are acceptable for passive components, they are normally far too large for opamp gains because the active parameters in a circuit must be expected to vary widely. Recall from the discussion in Chapter 2 that the opamp's gain-bandwidth product ω_t can be expected to vary by 30% to over 100% in processing, in addition to being dependent on operating conditions such as temperature and bias.

We will next derive some general formulas that follow from Eq. (12.8); they will be useful when computing sensitivities of the filter circuits in this book. We will again assume that T is a function of a parameter x, or of several parameters $x_1, x_2, x_3, \ldots$. If Y is independent of

x, then $S_x^Y = 0$, obviously, because the derivative is zero. If Y is proportional to x, that is, $Y = kx$, where k is a constant independent of x, then $S_x^Y = 1$:

$$Y = kx \quad \Rightarrow \quad S_x^Y = 1 \tag{12.14a}$$

Often, we will encounter the form $Y = kx^a$ where a and k are constants. From Eq. (12.8) we find directly $S_{x_1}^Y = a$:

$$Y = kx^a \quad \Rightarrow \quad S_x^Y = a \tag{12.14b}$$

If Y is a product of two (or more) functions, $Y = Y_1Y_2Y_3\cdots$, each of which depends on x, we have, with $\ln Y = \ln Y_1 + \ln Y_2 + \cdots$:

$$\frac{\partial \ln(Y_1Y_2Y_3\cdots)}{\partial \ln x} = \frac{\partial \ln Y_1}{\partial \ln x} + \frac{\partial \ln Y_2}{\partial \ln x} + \frac{\partial \ln Y_3}{\partial \ln x} + \cdots$$

The result is:

$$Y = Y_1Y_2Y_3\cdots \Rightarrow S_x^Y = S_x^{Y_1} + S_x^{Y_2} + \cdots \tag{12.14c}$$

i.e., the sensitivities of all functions Y_i add. In like manner, if Y is a quotient, $Y = Y_1/Y_2$, we find

$$Y = Y_1/Y_2 \Rightarrow S_x^Y = S_x^{Y_1} - S_x^{Y_2} \tag{12.14d}$$

from which it follows directly that

$$S_x^{1/Y} = \frac{\partial \ln(1/Y)}{\partial \ln x} = -\frac{\partial \ln Y}{\partial \ln x} = -S_x^Y \tag{12.14e}$$

A further very powerful relationship makes use of the chain rule of differentiation: if Y is a function of x, which in turn is a function of a variable t, we differentiate $\partial Y/\partial t = (\partial Y/\partial x)\times(\partial x/\partial t)$. The sensitivity is then computed as

$$Y = Y[x(t)] \quad \Rightarrow \quad S_t^Y = \frac{t}{Y}\frac{\partial Y}{\partial x}\frac{\partial x}{\partial t}\times\frac{x}{x} = \frac{x}{Y}\frac{\partial Y}{\partial x}\times\frac{t}{x}\frac{\partial x}{\partial t}$$

or

$$Y = Y[x(t)] \quad \Rightarrow \quad S_t^Y = S_x^Y S_t^x \tag{12.14f}$$

that is, we multiply the sensitivity of the function Y to the component x by the sensitivity of the component x to the variable t. This relationship is convenient to use, for example, when a filter parameter, say Q, is a function of a resistor R, which depends on temperature T. Then we compute $S_T^Q = S_R^Q S_T^R$. Or when we wish to determine how sensitive a transfer function $T(s)$ is to the gain-bandwidth product ω_t of an opamp A, we compute $S_{\omega_t}^{T(s)} = S_A^{T(s)} S_{\omega_t}^A$.

It is not sufficient, as a rule, to compute sensitivities to a single parameter, because the variations in all the circuit's components add to cause performance deviations. Thus, *single-parameter sensitivities* give only incomplete predictions of the variations to be expected. Conclusions based on single-parameter sensitivity results should be treated with caution. When a function depends on several elements, $Y = Y(x_1, x_2, \ldots, x_n)$, as all our filters do, and we wish to find the likely total change when *all* the elements are expected to vary, we make use again of differential calculus to find the total derivative. Assuming n components, we have

$$dY = dY\,(x_1, x_2, \ldots, x_n) = \frac{\partial Y}{\partial x_1}dx_1 + \frac{\partial Y}{\partial x_2}dx_2 + \cdots + \frac{\partial Y}{\partial x_n}dx_n$$

This equation is recast in the form

$$\frac{dY}{Y} = \frac{x_1}{Y}\frac{\partial Y}{\partial x_1}\frac{dx_1}{x_1} + \frac{x_2}{Y}\frac{\partial Y}{\partial x_2}\frac{dx_2}{x_2} + \cdots + \frac{x_n}{Y}\frac{\partial Y}{\partial x_n}\frac{dx_n}{x_n}$$
$$= S_{x_1}^{Y}\frac{dx_1}{x_1} + S_{x_2}^{Y}\frac{dx_2}{x_2} + \cdots + S_{x_n}^{Y}\frac{dx_n}{x_n} \quad (12.14g)$$

It says that the total percentage change of Y is the sum of all their respective sensitivities multiplied by the percentage changes of the components.

Equation (12.14g) gives a quick estimate of the expected circuit performance. More accurate *multiparameter sensitivity* results may be obtained from a *Monte Carlo* analysis. In Monte Carlo analysis we vary most or all components of a circuit randomly over a specified tolerance range, such as 1%, with given statistical distributions, such as uniform or Gaussian, to simulate the manufacturing process. We then perform analyses of the circuit for its response for each set of element values. Clearly, computer analysis with appropriate circuit-simulation software is necessary for that purpose because normally several hundred analyses are required to yield valid statistical results. The plots in Fig. 12.1 were obtained in this manner.

EXAMPLE 12.2

Calculate the sensitivity expressions for pole-frequency and quality factor of the *RLC* lowpass filter in Fig. 12.7.

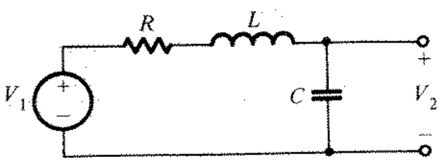


Figure 12.7 Passive *LC* lowpass filter.

Solution

The transfer function is

$$T(s) = \frac{V_2}{V_1} = \frac{1}{s^2LC + sCR + 1} = \frac{1}{LC}\frac{1}{s^2 + sR/L + 1/(LC)}$$

Clearly, we have

$$\omega_0 = \frac{1}{\sqrt{LC}} = L^{-1/2}C^{-1/2} \quad \text{and} \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}} = R^{-1}L^{1/2}C^{-1/2} \quad (12.15)$$

We obtain immediately from Eq. (12.14b)

$$S_L^{\omega_0} = -\frac{1}{2}, \quad S_C^{\omega_0} = -\frac{1}{2}, \quad S_R^{\omega_0} = 0$$

and

$$S_L^{Q} = +\frac{1}{2}, \quad S_C^{Q} = -\frac{1}{2}, \quad S_R^{Q} = -1$$

This analysis shows that a +1% change in L or C causes a 0.5% change in ω_0 and Q, and changing R by 1% causes no change in ω_0 but a −1% change in Q. The sign in each case indicates whether the change is increasing or decreasing. These sensitivities are considered low. The circuit is an example of an LC ladder filter; we shall find out in Chapter 13 that LC ladders in general tend to have very low sensitivities.

The total changes are computed with Eq. (12.14g) as

$$\frac{d\omega_0}{\omega_0} = S_L^{\omega_0}\frac{dL}{L} + S_C^{\omega_0}\frac{dC}{C} + S_R^{\omega_0}\frac{dR}{R} = -\frac{1}{2}\left(\frac{dL}{L} + \frac{dC}{C}\right) \quad (12.16a)$$

$$\frac{dQ}{Q} = S_L^{Q}\frac{dL}{L} + S_C^{Q}\frac{dC}{C} + S_R^{Q}\frac{dR}{R} = \frac{1}{2}\left(\frac{dL}{L} - \frac{dC}{C}\right) - \frac{dR}{R} \quad (12.16b)$$

We still make an important observation. If the relative changes in L and C are in the same direction, they add to changes in ω_0, but subtract in their contribution to changes in Q. This conclusion is intuitively obvious from Eq. (12.15): if both L and C increase, their effects on ω_0 add because ω_0 depends on a *product* of the two elements; but the errors tend to cancel in Q, which depends on a *ratio* of the components. An increase in R, i.e., larger losses, results in a decrease in Q, but does not affect ω_0.

The explanation at the end of Example 12.2 may need a more formal justification: Consider two components in a circuit, and let us take resistors R_1 and R_2 to be specific. If these components suffer tolerances ΔR_1 and ΔR_2, the elements in the circuit are not the nominal values R_{10} and R_{20}, but rather

$$R_1 = R_{10} + \Delta R_1 = R_{10}\left(1 + \frac{\Delta R_1}{R_{10}}\right) \quad \text{and} \quad R_2 = R_{20} + \Delta R_2 = R_{20}\left(1 + \frac{\Delta R_2}{R_{20}}\right)$$

where $\Delta R_1/R_{10}$ and $\Delta R_2/R_{20}$ are the relative errors. If a circuit is now a function of the *product* of the two components, the dependence is on

$$R_1 \times R_2 = R_{10}R_{20}\left(1 + \frac{\Delta R_1}{R_{10}}\right)\left(1 + \frac{\Delta R_2}{R_{20}}\right) = R_{10}R_{20}\left(1 + \frac{\Delta R_1}{R_{10}} + \frac{\Delta R_2}{R_{20}} + \frac{\Delta R_1}{R_{10}}\frac{\Delta R_2}{R_{20}}\right)$$

or, after neglecting the product of the errors as small,

$$R_1 \times R_2 \approx R_{10}R_{20}\left(1 + \frac{\Delta R_1}{R_{10}} + \frac{\Delta R_2}{R_{20}}\right) \quad (12.17a)$$

We notice that for a dependence on a *product*, the errors *add*. This is the situation with L and C in ω_0 in Eq. (12.15) and is reflected in the plus sign in Eq. (12.16a) between