Exercises for Tutorial 5: Interconnect

1. Consider the circuit shown in Figure 11. A voltage source with a $Z_s = 200\ \Omega$ output impedance driveas a transmission line with a characteristic impedance of $Z_0 = 50\ \Omega$ terminated in a load impedance of $Z_L = 12.5\ \Omega$. The transmission line has a propagation delay of $t_d = 2\ ns$. For a unit step on the voltage source, how long does it take before the reflected wave on the line has an amplitude less than $10\ mV$?

First calculate the relflection coefficients:

$$\Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} = \frac{150}{250} = \frac{3}{5} = 0.6$$
$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = -\frac{37.5}{62.5} = -\frac{3}{5} = -0.6$$

Time [ns]	Calculation
0	$V_{S0} = 0.2\ V$
2	$V_{L0} = V_{S0} + \Gamma_L V_{S0} = 0.2 - 0.6 \times 0.2 = 0.08\ V$
4	$V_{S1} = V_{S0} + \Gamma_L V_{S0} + \Gamma_S \Gamma_L V_{S0} = V_{L0} + \Gamma_S \Gamma_L V_{S0} = 0.08 - 0.008 = 0.072\ V$
6	$V_{L1} = 0.2912\ V$
8	$V_{S2} = 0.31713\ V$
10	$V_{L2} = 0.301568\ V$
12	$V_{S3} = 0.301568 - 0.0093312 = 0.2922368\ V$

The reflection is less than 10 mV after 12 ns.

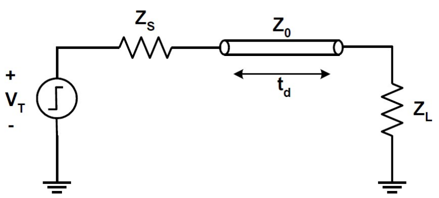

Figure 11 A voltage source driving a terminated transmission line.

2. If a lossless transmission line of characteristic impedance Z_0 and length l, driven sinusoidally, is terminated in an arbitrary impedance Z_L, show that the impedance at the drive end is $Z(l) = Z_0 \frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}$ where $\beta = \omega\sqrt{LC}$. The term β determines the rate at which the phase changes along the line and is equal to $\frac{2\pi}{\lambda}$, where λ is the wavelength of the excitation frequency.

Lossless transmission line: $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega\sqrt{LC} = j\beta$ since $R = G = 0$.
At point x on the line the voltage is a combination of the incident and reflected waves.

$$V(x) = V_i e^{-j\beta x} + V_r e^{j\beta x}$$
$$I(x) = I_i e^{-j\beta} \;\; + I_r e^{j\beta x}$$

Then we have:

$$Z(x) = \frac{V(x)}{I(x)} = \frac{V_i}{I_i}\frac{e^{(-j\beta x)\left(1+\frac{V_r}{V_i}e^{j2\beta x}\right)}}{e^{(-j\beta x)\left(1-\frac{I_r}{I_i}e^{j2\beta x}\right)}} = Z_0\frac{1+\Gamma_L e^{j2\beta x}}{1-\Gamma_L e^{j2\beta x}}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \to Z(x) = Z_0\frac{Z_0 + Z_L + (Z_L - Z_0)e^{j2\beta x}}{Z_0 + Z_L - (Z_L - Z_0)e^{j2\beta x}} = Z_0\frac{Z_L\left(1+e^{j2\beta x}\right) + Z_0\left(1-e^{j2\beta x}\right)}{Z_0\left(1+e^{j2\beta x}\right) + Z_L\left(1-e^{j2\beta x}\right)}$$

$$1 + e^{j2\beta x} = e^{j\beta x}\left(e^{-j\beta x} + e^{j\beta x}\right) = 2e^{j\beta x}\frac{\left(e^{j\beta x} + e^{-j\beta x}\right)}{2} = 2e^{j\beta x}\cos(\beta x)$$
$$1 - e^{j2\beta x} = e^{j\beta x}\left(e^{-j\beta} \;\; - e^{j\beta x}\right) = -2e^{j\beta x}\frac{\left(e^{j\beta x} - e^{-j\beta x}\right)}{2j} = -2e^{j\beta x} j\sin(\beta x)$$

$$Z(x) = Z_0\frac{2e^{j\beta x}(Z_L\cos(\beta x) - jZ_0\sin(\beta x))}{2e^{j\beta x}(Z_0\cos(\beta x) - jZ_L\sin(\beta x))} = Z_0\frac{Z_L - jZ_0\frac{\sin(\beta x)}{\cos(\beta x)}}{Z_0 - jZ_L\frac{\sin(\beta x)}{\cos(\beta x)}}$$

$$Z(x) = Z_0\frac{Z_L - jZ_0\tan(\beta x)}{Z_0 - jZ_L\tan(\beta x)}$$

At the drive end $x = -l$ so $Z(-l\,) = Z_0\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}$

3. In problem 2, what is the input impedance as a function of load impedance when the length of the line is $\frac{\lambda}{4}$. For this case what is the input impedance when the line is shorted? What is the input impedance when the line is open?

$l = \frac{\lambda}{4}$ and $\beta = \frac{2\pi}{\lambda}$ gives $\tan(\beta l) = \tan\left(\frac{2\pi}{4}\right) \to \infty$

So: $Z\left(-\frac{\lambda}{4}\right) = Z_0\frac{jZ_0}{jZ_L} = \frac{Z_0^2}{Z_L}$

If the line is shorted then $Z_L = 0$ so $Z\left(-\frac{\lambda}{4}\right) \to \infty$

If the line is open then $Z_L = \infty$ so $Z\left(-\frac{\lambda}{4}\right) \to 0$

4. Estimate the delay of the wire with a high frequency $Z_0 = 50\ \Omega$, resistance $r = 14\ k\Omega\ /m$ and dielectric constant $\varepsilon_r = 4$.

   a) Length $d = 1\ mm$.

$$R = rl = 14e3 \times 1e-3 = 14$$
$$e^{-\frac{R}{2Z_0}} \approx 0.87 > \frac{1}{2} \rightarrow LC\ line$$
$$td = \frac{d}{v} = d\frac{\sqrt{\epsilon_r\mu_r}}{C_0} = \frac{d\sqrt{\epsilon_r}}{C_0} = \frac{2e-3}{3e8} = \frac{2}{3} \times 10^{-1} \quad \approx 6.67\ ps$$

   b) Length $d = 25\ mm$.

$$R = rl = 14e3 \times 25e-3 = 350$$
$$e^{-\frac{R}{2Z_0}} \approx 0.03 < \frac{1}{2} \rightarrow RC\ line$$
$$td \quad 0.38\ d^2rC$$

We need to find C:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R + j\omega L}{j\omega C}} \rightarrow$$
$$j\omega C = \frac{R + j\omega L}{Z_0^2} \rightarrow C = \frac{\frac{R}{j\omega} + L}{Z_0^2} \approx \frac{L}{Z_0^2}\ for\ high\ frequencies\ (1)$$

Now we have C but have to find L:

$$v = \frac{1}{\sqrt{LC}} = \frac{C_0}{\sqrt{\epsilon_r}} \rightarrow L = \frac{\epsilon_r}{C_0^2}\frac{1}{C}\ (2)$$

(2) in (1):

$$C = \frac{1}{Z_0^2}\frac{\epsilon_r}{C_0^2}\frac{1}{C} = \frac{\sqrt{\epsilon_r}}{C_0}\frac{1}{Z_0}$$
$$t_d = 0.38\ d^2r\frac{\sqrt{\epsilon_r}}{C_0}\frac{1}{Z_0} = 0.38 \times 0.025^2 \times 14 \times 10^3 \times \frac{\sqrt{4}}{3 \times 10^8} \times \frac{1}{50} \approx 4.43 \times 10^{-10}$$
$$td \quad 443\ ps$$

5. An RC-wire with length of d is divided into N equal-length sections and between each section repeaters are inserted as shown in Figure 12. To get the optimal delay through the wire, what section length should be chosen? How many repeaters are needed? Assume that the propagation delay of each repeater is t_p and the wire resistance and wire capacitance per unit length is r and c, respectively.

Section length $d_s = \frac{d}{N}$

Delay of a section $t_{ds} = 0.38\ d_s^2rc \quad 0.38\frac{d^2}{N^2}rc$

Delay of a repeater t_p

Total delay of line $t_d = Nt_p + Nt_{ds} = Nt_p + 0.38\frac{d^2}{N}rc$

Find optimal N, derive t_d with respect to N and set to 0.

$$\frac{dt_d}{dN} = t_p - 0.38\frac{d^2}{N^2}rc \quad 0 \rightarrow N_{opt} = \sqrt{\frac{0.38rc}{t_p}}d$$

Optimal section length $\frac{d_s}{N_{opt}} = \sqrt{\frac{t_p}{0.38rc}}$ Independent of d!

Optimal number of repeaters $ceil(N_{opt}) = \left\lceil d\sqrt{\frac{0.38rc}{t_p}} \right\rceil$

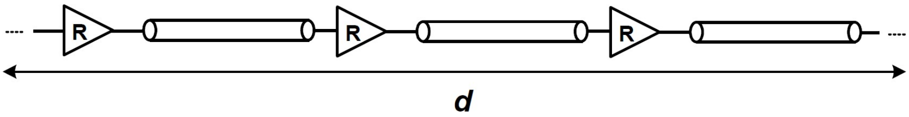

Figure 12 Inserting repeaters into an RC line.